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Inference on inverted exponentiated Rayleigh data from accelerated life testing with hybrid censoring

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ABSTRACT

This paper addresses the problem of estimating unknown parameters of the inverted exponentiated Rayleigh distribution within the context of accelerated life testing. We consider lifetime data observed through step-stress and type-I hybrid censoring, and incorporate the cumulative exposure model assumptions to establish connections between the distribution at various stress levels. We then write the associated likelihood function based on the observed data and derive maximum likelihood estimators for the distribution's unknown parameters. Furthermore, employing a Bayesian approach, we initially adopt gamma priors and compute posterior distributions for the parameters. These posterior distributions are then utilized to calculate Bayesian estimates using the squared error loss function. To assess the performance of maximum likelihood and Bayesian estimates, we conduct a simulation study under various scenarios, considering both non-informative and informative priors. We also evaluate interval estimates and coverage percentages under both classical and Bayesian approaches. Finally, for illustrative purposes, we analyze two real data sets, demonstrating the practical application of our proposed methodology.

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Accelerated life testing; cumulative exposure model; Bayesian estimates; hybrid censoring; inverted exponentiated Rayleigh distribution; Metropolis-Hastings algorithm

1. Introduction

In the areas of life testing and reliability analysis, obtaining failure lifetime data under typical operating conditions can be time-consuming, especially for highly reliable units. The accelerated life test (ALT) addresses this challenge by subjecting units to higher stress levels than normal to induce rapid failures. ALT experiments can be conducted in three main ways based on the applied stress: (i) constant stress, (ii) step-stress and (iii) progressive stress. In constant stress ALT, all units are exposed to a consistent stress level throughout the test. However, this approach may result in too few failures if the stress level is too low, or excessive costs if the stress level is too high. Step-stress ALT involves increasing stress levels at specified times to achieve an adequate number of failures within a shorter period, while progressive stress ALT continuously increases stress levels over time. Progressive stress testing generally requires more resources compared to constant and step-stress methods. The choice of ALT depends on factors such as the units being tested, available resources and budget.

Existing literature, including works by W. Nelson (1980) and W. B. Nelson (2009), provides comprehensive coverage of ALT methodologies. Despite this, ALTs may face limitations when

experiments are terminated, either intentionally or unintentionally, leading to censored data where only partial information about failures is available.

In life testing, reliability and survival analysis, censoring schemes such as type-I and type-II are commonly used. Type-I censoring ends experiments at a predetermined time, while type-II censoring concludes them after a specific number of failures. Type-I censoring provides a fixed endpoint but may yield insufficient data if failures do not occur by the set time, affecting the robustness of the statistical analysis. Conversely, type-II censoring ensures a minimum amount of failure data but can be resource-intensive if failures take longer to occur. To address these issues, hybrid censoring combines aspects of both schemes. This approach, proposed by Epstein (1954) and utilized in standards like MIL-STD-781-C Standard (1977), is discussed extensively in Balakrishnan et al. (2023). Hybrid censoring integrates type-I and type-II criteria to provide a more comprehensive assessment, capturing both time-based and failure-based data. This integration with ALT enhances supply chain management, reduces unexpected failures, shortens time-to-market, lowers development costs and improves product quality, thereby supporting sustainable business growth and competitive advantage. For example, in vehicle component lifespan testing, the focus may be on failure or reaching a predefined time/mileage under accelerated driving conditions. Similarly, in pharmaceutical stability testing, some drug samples are tested until a predefined time, while others are tested until degradation or failure. In existing literature, lifetime data from ALTs with the hybrid censoring has been analysed by Balakrishnan and Xie (2007), Lin et al. (2020) and Yadav et al. (2023). One may also refer to the work of Kotb and El-Din (2020), Nassar et al. (2021), Kim and Sung (2023), Wu et al. (2023) and Dutta et al. (2023) for statistical analysis of lifetime data observed from ALTs with different censoring schemes.

This paper focuses on accelerated lifetime data following an inverted exponentiated Rayleigh (IER) distribution with hybrid censoring. The IER distribution, introduced by Ghitany et al. (2014), is part of the inverse exponentiated class of distributions and is particularly useful for modeling data with non-monotone hazard rates. Such rates are crucial in reliability engineering, as they describe failure rates that do not follow a simple increasing or decreasing trend over time. For instance, many electronic devices show high initial failure rates due to manufacturing defects, which stabilize over time. Similarly, systems under fluctuating stresses or cyclic loading exhibit varying failure rates. The IER distribution, with its ability to model complex hazard rate behaviors, offers a valuable alternative to traditional distributions like the log-normal and inverse Weibull. Its application in ALT and hybrid censoring methodologies enhances the accuracy and depth of reliability analyses, leading to improved product reliability and efficiency. The probability density function (PDF) and the cumulative distribution function (CDF) of a random variable T following the IER(α, θ) distribution are, respectively, given by

$$f(t, \theta, \alpha) = 2\alpha\theta t^{-3} e^{-\frac{\theta}{t^2}} (1 - e^{-\frac{\theta}{t^2}})^{\alpha-1}, \quad t > 0, \alpha > 0, \theta > 0, \quad (1)$$

$$\text{and } F(t, \theta, \alpha) = 1 - (1 - e^{-\frac{\theta}{t^2}})^{\alpha}, \quad (2)$$

Here, α and θ represent the shape and scale parameters, respectively. Notably, this distribution simplifies to an inverted Rayleigh distribution when $\alpha = 1$. In life testing and reliability analysis, estimating the unknown parameters of a statistical distribution is crucial for understanding and interpreting data, making predictions and guiding decisions. Parameter estimates derived from a sample allow us to infer characteristics about the entire population. The corresponding interval estimates provide a range of possible values, reflecting the uncertainty in these predictions. The IER distribution has recently garnered attention due to its non-monotone hazard rate. Researchers such as Rastogi and Tripathi (2014), Kayal et al. (2018), Maurya et al. (2019), Panahi and Moradi (2020) and Wang et al. (2024) have provided insights into estimating its parameters, particularly for data observed under normal operating conditions and various censoring schemes. Recent studies by Asadi and Panahi (2022) focused on stress – strength reliability under ALTs and applied these results to coating processes, while Ma et al. (2023) examined reliability stress-strength parameters

using maximum likelihood and pivotal-based inference. However, our review reveals a notable gap in the literature regarding statistical inference from both classical and Bayesian perspectives for lifetime data following the IER distribution observed under ALT with step-stress and hybrid censoring. This gap motivated us to address this issue in our current paper. By employing both classical and Bayesian methods, we aim to provide robust statistical inferences through point and interval estimates for the unknown parameters of the IER distribution, thereby contributing to the existing literature on reliability analysis.

The remainder of the paper is organized as follows: [Section 2](#) covers the collection of lifetime data from the IER distribution when ALTs are conducted under step-stress with hybrid censoring. [Section 3](#) discusses the computation of maximum likelihood estimates for the unknown parameters and their associated interval estimates. [Section 4](#) focuses on Bayesian estimates. [Section 5](#) presents a simulation study and illustrates the proposed methodology using a numerical example. Two real datasets are analyzed in [Section 6](#), and the paper concludes in [Section 7](#).

2. Model assumptions

This section outlines the model assumptions, procedures for conducting ALT under step-stress conditions, and lifetime data collection under type-I hybrid censoring. Consider a life test experiment where n units are subjected to two stress levels: an initial stress level x_1 and a higher stress level x_2 . The lifetimes of units under x_1 follow an $\text{IER}(\alpha, \theta_1)$ distribution, while those under x_2 follow an $\text{IER}(\alpha, \theta_2)$ distribution. We assume the shape parameter $\alpha > 0$ is invariant to stress levels, while the scale parameters θ_i are expressed using a linear link function: $\theta_i = a_0 + a_1 x_i$, $0 \leq x_i \leq 1$. To model the impact of varying stress levels, we adopt the cumulative exposure model (CEM), which assumes that damage accumulates over time as units are exposed to different stress levels. Each stress level contributes incrementally to the total damage, eventually causing failure. For further details on the CEM, one may refer to Miller and Nelson (1983). The CDF for the lifetimes using CEM is given by

$$G(t; \theta, \alpha) = \begin{cases} 1 - (1 - e^{-\frac{\theta_1}{t}})^{\alpha}, & 0 < t \leq \tau_1, \\ 1 - (1 - e^{-\frac{\theta_2}{(t-\tau_1+\delta)^2}})^{\alpha}, & \tau_1 < t < \infty, \end{cases} \quad (3)$$

where τ_1 represents the stress changing time, and $\delta = \sqrt{\frac{\theta_2}{\theta_1}}\tau_1$ is assumed to be equivalent start time in the second step of stress, which is the solution to the equation $F(\delta|\theta_2, \alpha) = F(\tau_1|\theta_1, \alpha)$. Therefore, the PDF of a test unit, say $g(t; \theta, \alpha)$, can be obtained by taking the derivative of the corresponding CDF function $G(t; \theta, \alpha)$ given by (3).

Under type-I hybrid censoring, the experimenter pre-fixes the stress-changing time τ_1 and decides to terminate the experiment either when a pre-fixed m number of failures are observed or when the time τ_2 is reached, whichever occurs earlier. By default, $\tau_1 < \tau_2$, as the stress-changing time precedes the termination time. Under this setup, the experiment terminates at a random time $T = \min(\tau_2, t_m)$, where t_m denotes the failure time of the m -th unit. A random number of failures, say d , may occur by the time τ_2 . The outcome of the experiment is represented by r observed failure times, $t = (t_1, t_2, \dots, t_r)$. Specifically: If $T = \tau_2$, corresponding to type-I censoring, $r = d$, and if $T = t_m$, corresponding to type-II censoring, $r = m$. In fact, the observed failure lifetimes can fall into one of the following scenarios based on their relation to the stress-changing time τ_1 . Case (i): All r failures occur before the stress-changing time τ_1 , i.e. $(t_1, t_2, \dots, t_r \leq \tau_1 < \tau_2)$. In this case: either the pre-fixed m failures occur before τ_1 , leading to termination at $T = t_m$, or some d failures occur before τ_1 , and no failures occur afterward, with the experiment terminating at $T = \tau_2$. Case (ii): All r failures occur after the stress-changing time τ_1 , i.e. $(\tau_1 < t_1, t_2, \dots, t_r \leq \tau_2)$. Here, r may correspond to either type-I or type-II censoring, depending on whether $T = \tau_2$ or $T = t_m$. Case (iii): Some n_1 failures occur before τ_1 , and the remaining $n_2 = (r - n_1)$ failures occur after τ_1 . In this case, $(t_1, t_2, \dots, t_{n_1} \leq \tau_1 < t_{n_1+1}, \dots, t_r \leq \tau_2)$. It is worth noting that data under constant stress can be

considered a special case of scenario (i), where τ_1 is chosen to be sufficiently large such that all failures occur before the stress change at τ_1 . Further, similar to type-I hybrid censoring, type-II hybrid censoring is defined by terminating the experiment at a random time $T = \max(\tau_2, t_m)$, with $\tau_1 < \tau_2$. Type-I hybrid censoring is optimal when time or budget constraints are a priority, as it provides a fixed endpoint for the experiment. In contrast, type-II hybrid censoring is more suitable when obtaining comprehensive failure data is the primary goal, even if it requires longer testing durations. In this work, we focus on type-I hybrid censoring. However, the methodology and all derived expressions are equally applicable to type-II hybrid censoring by considering $T = \max(\tau_2, t_m)$, with $\tau_1 < \tau_2$.

3. Maximum likelihood estimation

In this section, we analyze the data $t = (t_1, t_2, \dots, t_r)$, observed from the IER(α, θ) distribution under an ALT setup involving type-I hybrid censoring, as outlined in the previous section. The goal is to derive the maximum likelihood estimators (MLEs) for the unknown parameters of the IER distribution. The likelihood function for $(\theta_1, \theta_2, \alpha)$ given the observed data t can be expressed as follows

$$L(\theta_1, \theta_2, \alpha|t) = \prod_{i=1}^{n_1} g(t_i; \theta_1, \alpha) \prod_{i=1}^{n_2} g(t_{n_1+i}; \theta_2, \alpha) (1 - G(T; \theta_1, \theta_2, \alpha))^{n-r} \quad (4)$$

where $G(\cdot)$ denotes the CDF given by (3), and $g(t_i; \theta_1, \alpha)$ and $g(t_i; \theta_2, \alpha)$, respectively, denotes the PDF obtained by taking the derivative of the corresponding CDF based on $t < \tau_1$ or $t > \tau_1$. The parameters θ_1 and θ_2 correspond to the stress levels before and after the stress-changing time τ_1 , respectively. It is to be noticed that the likelihood function $L(\theta_1, \theta_2, \alpha|t)$ is subject to three distinct cases based on the observed failure times relative to the stress-changing time τ_1 . Case (i): When all the r failures occur before the stress-changing time τ_1 . In this case, the term $\prod_{i=1}^{n_2} g(t_{n_1+i}; \theta_2, \alpha)$ does not appear in the likelihood function since $n_2 = 0$, and therefore $n_1 = r$. Case (ii): When all the r failures occur after the stress-changing time τ_1 . In this case, the term $\prod_{i=1}^{n_1} g(t_i; \theta_1, \alpha)$ does not appear in the likelihood function since $n_1 = 0$, and therefore $n_2 = r$. In case (iii): when n_1 failures occur before τ_1 , and the remaining $n_2 = (r - n_1)$ failures occur after τ_1 . The full likelihood function as given in (4) is used. For consistency and completeness, we will consider the likelihood function as expressed in (4), with cases (i) and (ii) treated as special instances where the respective terms $\prod_{i=1}^{n_2} g(t_{n_1+i}; \theta_2, \alpha)$ or $\prod_{i=1}^{n_1} g(t_i; \theta_1, \alpha)$ are omitted. This approach streamlines the formulation and allows us to derive the log-likelihood function, $l = \ln L(\theta_1, \theta_2, \alpha|t)$, as follows

$$l = r \ln \alpha + n_1 \ln \theta_1 + n_2 \ln \theta_2 - \xi_1(\theta_1, \theta_2) - 3\xi_2(\theta_1, \theta_2) + (\alpha - 1)\xi_3(\theta_1, \theta_2) + \alpha(n - r)\xi_4(\theta_1, \theta_2), \quad (5)$$

$$\text{where } \xi_1(\theta_1, \theta_2) = \sum_{i=1}^{n_1} \theta_1 t_i^{-2} + \sum_{i=1}^{n_2} \theta_2 S_i^{-2},$$

$$\xi_2(\theta_1, \theta_2) = \sum_{i=1}^{n_1} \ln t_i + \sum_{i=1}^{n_2} \ln S_i,$$

$$\xi_3(\theta_1, \theta_2) = \sum_{i=1}^{n_1} \ln(1 - e^{-\theta_1 t_i^{-2}}) + \sum_{i=1}^{n_2} \ln(1 - e^{-\theta_2 S_i^{-2}}),$$

$$\xi_4(\theta_1, \theta_2) = \ln(1 - e^{-\theta_2 S_T^{-2}}),$$

$$S_i = (t_{n_1+i} + \delta - \tau_1), i = 1, 2 \dots n_2,$$

and $S_T = (T + \delta - \tau_1)$. Now, on differentiating the log-likelihood function, given by (5), with respect to θ_1, θ_2 and α and equating them to zero, we get the following equations

$$\frac{\partial l}{\partial \theta_1} = \frac{n_1}{\theta_1} - \xi_1^{\theta_1}(\theta_1, \theta_2) - 3\xi_2^{\theta_1}(\theta_1, \theta_2) + (\alpha - 1)\xi_3^{\theta_1}(\theta_1, \theta_2) + \alpha(n - r)\xi_4^{\theta_1}(\theta_1, \theta_2) = 0,$$

$$\frac{\partial l}{\partial \theta_2} = \frac{n_2}{\theta_2} - \xi_1^{\theta_2}(a_0, a_1) - 3\xi_2^{\theta_2}(\theta_1, \theta_2) + (\alpha - 1)\xi_3^{\theta_2}(\theta_1, \theta_2) + \alpha(n - r)\xi_4^{\theta_2}(\theta_1, \theta_2) = 0,$$

$$\frac{\partial l}{\partial \alpha} = \frac{r}{\alpha} + \xi_3(\theta_1, \theta_2) + (n - r)\xi_4(\theta_1, \theta_2) = 0,$$

where

$$\xi_1^{\theta_1}(\theta_1, \theta_2) = \frac{\partial \xi_1(\theta_1, \theta_2)}{\partial \theta_1} = \sum_{i=1}^{n_1} t_i^{-2} + \theta_2 \theta_1^{-1} \delta \sum_{i=1}^{n_2} S_i^{-3},$$

$$\xi_2^{\theta_1}(\theta_1, \theta_2) = \frac{\partial \xi_2(\theta_1, \theta_2)}{\partial \theta_1} = -\frac{1}{2} \theta_1^{-1} \delta \sum_{i=1}^{n_2} S_i^{-1},$$

$$\xi_3^{\theta_1}(\theta_1, \theta_2) = \frac{\partial \xi_3(\theta_1, \theta_2)}{\partial \theta_1} = \sum_{i=1}^{n_1} A_{1i} t_i^{-2} + \delta \theta_2 \theta_1^{-1} \sum_{i=1}^{n_2} A_{2i} S_i^{-3},$$

$$\xi_4^{\theta_1}(\theta_1, \theta_2) = \frac{\partial \xi_4(\theta_1, \theta_2)}{\partial \theta_1} = \delta \theta_2 \theta_1^{-1} A_{2T} S_T^{-3},$$

$$\xi_1^{\theta_2}(\theta_1, \theta_2) = \frac{\partial \xi_1(\theta_1, \theta_2)}{\partial \theta_2} = \sum_{i=1}^{n_2} S_i^{-2} (\delta S_i^{-1} + 1),$$

$$\xi_2^{\theta_2}(\theta_1, \theta_2) = \frac{\partial \xi_2(\theta_1, \theta_2)}{\partial \theta_2} = \sum_{i=1}^{n_2} S_i^{-1} B,$$

$$\xi_3^{\theta_2}(\theta_1, \theta_2) = \frac{\partial \xi_3(\theta_1, \theta_2)}{\partial \theta_2} = \sum_{i=1}^{n_2} A_{2i} S_i^{-2} (S_i^{-1} \delta + 1),$$

$$\xi_4^{\theta_2}(\theta_1, \theta_2) = \frac{\partial \xi_4(\theta_1, \theta_2)}{\partial \theta_2} = A_{2T} S_T^{-2} (S_T^{-1} \delta + 1),$$

$B = \frac{\partial \delta}{\partial \theta_2} = -\frac{1}{2\theta_1} \left(\frac{\theta_2}{\theta_1}\right)^{-\frac{1}{2}} \tau_1$, $A_{1i} = \sum_{i=1}^{n_1} (1 - e^{-\theta_1 t_i^{-2}})^{-1} e^{-\theta_1 t_i^{-2}}$, $A_{2i} = \sum_{i=1}^{n_2} (1 - e^{-\theta_2 S_i^{-2}})^{-1} e^{-\theta_2 S_i^{-2}}$ and $A_{2T} = (1 - e^{-\theta_2 S_T^{-2}})^{-1} e^{-\theta_2 S_T^{-2}}$. We observe that the aforementioned equations are non-linear in nature. Therefore, to solve these equations, we must use some numerical iterative methods such as Newton-Raphson (NR) or some generalized version of NR method to estimate the unknown parameters through iterations. The convergence can be achieved on getting $|\theta_1^{(k+1)} - \theta_1^{(k)}| + |\theta_2^{(k+1)} - \theta_2^{(k)}| + |\alpha^{(k+1)} - \alpha^{(k)}| \leq \varepsilon$, where ε is a given small fixed value. Here, β denotes as a vector of unknown parameters, that is, $\beta = (\theta_1, \theta_2, \alpha)$, and $\beta^{(k)}$

represents the value of the parameter at the k -th iteration. We denote the desired maximum likelihood estimate (MLE) of β as $\hat{\beta}$.

3.1. Interval estimation

The objective of this section is to construct an interval estimate that is likely to contain the true value of an unknown population parameter with a certain level of confidence. The observed Fisher information matrix (FIM), denoted as $I(\beta)$, is given by

$$I(\beta) = - \left\{ \frac{\partial^2 l}{\partial \beta_i \partial \beta_j} \right\} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The elements of FIM are calculated as follows

$$I_{11} = -\frac{n_1}{\theta_1^2} - \xi_1^{\theta_1 \theta_1}(\theta_1, \theta_2) - 3\xi_2^{\theta_1 \theta_1}(\theta_1, \theta_2) + (\alpha - 1)\xi_3^{\theta_1 \theta_1}(\theta_1, \theta_2) + \alpha(n - r)\xi_4^{\theta_1 \theta_1}(\theta_1, \theta_2),$$

$$I_{22} = -\frac{n_2}{\theta_2^2} - \xi_1^{\theta_2 \theta_2}(\theta_1, \theta_2) - 3\xi_2^{\theta_2 \theta_2}(\theta_1, \theta_2) + (\alpha - 1)\xi_3^{\theta_2 \theta_2}(\theta_1, \theta_2) + \alpha(n - r)\xi_4^{\theta_2 \theta_2}(\theta_1, \theta_2),$$

$$I_{12} = -\xi_1^{\theta_2 \theta_1}(\theta_1, \theta_1) - 3\xi_2^{\theta_2 \theta_1}(\theta_1, \theta_2) + (\alpha - 1)\xi_3^{\theta_2 \theta_1}(\theta_1, \theta_2) + \alpha(n - r)\xi_4^{\theta_2 \theta_1}(\theta_1, \theta_2),$$

$$I_{13} = \xi_3^{\theta_1}(\theta_1, \theta_2) + (n - r)\xi_4^{\theta_1}(\theta_1, \theta_2),$$

$$I_{23} = \xi_3^{\theta_2}(\theta_1, \theta_2) + (n - r)\xi_4^{\theta_2}(\theta_1, \theta_1),$$

$$I_{33} = -\frac{r}{\alpha^2}.$$

The details regarding the involved expressions in second order partial derivatives are reported in the [Appendix](#). Next, to compute the asymptotic confidence interval estimates for the unknown model parameters based on the asymptotic distribution of the MLE, we take

$$I^{-1}(\hat{\beta}) = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}.$$

Now, based on the normal approximation of the MLE and Slutsky's theorem, we can show that the distribution of the pivotal quantity $Z_{\beta_h} = (\hat{\beta}_h - \beta_h) / \sqrt{V_{hh}}$, $h = 1, 2, 3$, converges in distribution to the standard normal distribution. Therefore, a two-sided $100(1 - \rho)\%$ asymptotic confidence interval for β_h is given by $\hat{\beta}_h \pm z_{1-\rho/2} \sqrt{V_{hh}}$, where $z_{1-\rho/2}$ is the upper $\rho/2$ percentile of the standard normal distribution.

4. Bayesian estimation

The objective of this section is to compute Bayesian estimates under the squared error loss function. If $\hat{\beta}$ is an estimator for the unknown parameter β , then the squared error loss function is defined as $L(\beta, \hat{\beta}) = (\hat{\beta} - \beta)^2$. This loss function assigns equal weights to underestimation as well as overestimation. In the existing literature, Rastogi and Tripathi (2014), Kayal et al. (2018), and recently Panahi and Moradi (2020) considered the squared error loss function and suggested independent gamma priors for the parameters of IER distribution when lifetime data are observed under normal

operating conditions in the presence of different censoring schemes. In fact, gamma priors have been utilized in Bayesian inference for decades, offering a robust and intuitive framework built on a rich legacy of statistical research. One of the main reasons for their popularity is their conjugate nature, which often simplifies the mathematical and computational process by allowing the analytical derivation of posterior distributions. This mathematical simplicity is complemented by the flexibility of the gamma distribution's shape and scale parameters, making it ideal for encoding prior beliefs about underlying population parameters. We next consider independently distributed gamma priors for θ_1 , θ_2 and α , such as

$$\pi(\theta_1|\mu_1, \nu_1) \propto \theta_1^{\mu_1-1} e^{-\nu_1\theta_1}, \mu_1 > 0, \nu_1 > 0,$$

$$\pi(\theta_2|\mu_2, \nu_2) \propto \theta_2^{\mu_2-1} e^{-\nu_2\theta_2}, \mu_2 > 0, \nu_2 > 0,$$

$$\pi(\alpha|\mu_3, \nu_3) \propto \alpha^{\mu_3-1} e^{-\nu_3\alpha}, \mu_3 > 0, \nu_3 > 0,$$

where μ_i and ν_i , $i = 1, 2, 3$, are the hyper-parameters responsible for reflecting the prior knowledge about the unknown parameters. Notice that the considered priors reduce to non-informative priors when the hyper-parameter values are considered approaching zeros. Now using the joint prior $\pi(\theta_1, \theta_2, \theta_3) = \pi(\theta_1|\mu_1, \nu_1)\pi(\theta_2|\mu_2, \nu_2)\pi(\alpha|\mu_3, \nu_3)$, the subsequent posterior distribution can be expressed in the following form

$$\begin{aligned} \pi(\theta_1, \theta_2, \alpha|t) &= \frac{\pi(\theta_1|\mu_1, \nu_1)\pi(\theta_2|\mu_2, \nu_2)\pi(\alpha|\mu_3, \nu_3)L(\theta_1, \theta_2, \alpha|t)}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\theta_1|\mu_1, \nu_1)\pi(\theta_2|\mu_2, \nu_2)\pi(\alpha|\mu_3, \nu_3)L(\theta_1, \theta_2, \alpha|t) d\theta_1 d\theta_2 d\alpha}, \\ &\propto K^{-1} \psi_1(\theta_1, \theta_2, \alpha) e^{-\psi_2(\theta_1, \theta_2, \alpha)}, \end{aligned}$$

where

$$K = \int_0^\infty \int_0^\infty \int_0^\infty \psi_1(\theta_1, \theta_2, \alpha) e^{-\psi_2(\theta_1, \theta_2, \alpha)} d\theta_1 d\theta_2 d\alpha,$$

$$\psi_1(\theta_1, \theta_2, \alpha) = \nu_1\theta_1 + \nu_2\theta_2 + \nu_3\alpha + \theta_1 \sum_{i=1}^{n_1} t_i^{-2} + \theta_2 \sum_{i=1}^{n_2} S_i^{-2},$$

$$\begin{aligned} \psi_2(\theta_1, \theta_2, \alpha) &= \alpha^{r+\mu_3-1} \theta_1^{n_1+\mu_1-1} \theta_2^{n_2+\mu_2-1} \prod_{i=1}^{n_1} t_i^{-3} (1 - e^{-\theta_1 t_i^{-2}})^{\alpha-1} \prod_{i=1}^{n_2} S_i^{-3} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} \\ &\quad \times (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}. \end{aligned}$$

In the existing literature, Lindley (1980) and Tierney and Kadane (1986) methods are extensively used to compute Bayesian estimates. Furthermore, when the posterior distribution turns out to follow a well-known distribution or a product of well-known distributions, samples can be drawn using importance sampling or Gibbs sampling to compute Bayesian estimates. However, instead of proceeding with these traditional methods, we focus on finding the posterior marginal distributions of the parameters and then implement the Metropolis-Hastings algorithm for computation purposes.

4.1. Metropolis-Hastings algorithm

The Metropolis-Hastings (MH) algorithm is a Markov Chain Monte Carlo (MCMC) method used to generate samples from a probability distribution when direct sampling is difficult. In a Bayesian approach, posterior distributions often do not correspond to well-known distributions, making direct sampling challenging. In our case, the resulting posterior distributions are not well-known. Therefore, we first obtain the posterior marginal distributions of θ_1 , θ_2 , and α , and then use the MH algorithm to generate random samples from these distributions. We observe that the posterior marginal distributions have the following form:

$$\pi^*(\theta_1|\theta_2, \alpha) = \theta_1^{n_1+\mu_1-1} e^{-\theta_1 \sum_{i=1}^{n_1} t_i^{-2} - \theta_2 \sum_{i=1}^{n_2} S_i^{-2} - \nu_1 \theta_1} \prod_{i=1}^{n_1} (1 - e^{-\theta_1 t_i^{-2}})^{\alpha-1} \\ \times \prod_{i=1}^{n_2} S_i^{-3} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)},$$

$$\pi^*(\theta_2|\theta_1, \alpha) = \theta_2^{n_2+\mu_2-1} e^{-\theta_2 \sum_{i=1}^{n_2} S_i^{-2} - \nu_2 \theta_2} \prod_{i=1}^{n_2} S_i^{-3} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}, \quad (7)$$

$$\pi^*(\alpha|\theta_1, \theta_2) = \alpha^{r+\mu_3-1} e^{-\nu_3 \alpha} \prod_{i=1}^{n_1} (1 - e^{-\theta_1 t_i^{-2}})^{\alpha-1} \prod_{i=1}^{n_2} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}. \quad (8)$$

Algorithm 1 Metropolis-Hastings algorithm for sampling from posterior distributions

Step 1: Set the initial parameter values as $\theta_1^{(0)} = \hat{\theta}_1^{(0)}$, $\theta_2^{(0)} = \hat{\theta}_2^{(0)}$ and $\alpha^{(0)} = \hat{\alpha}^{(0)}$.

Step 2: Set the iteration index $j = 1$.

Step 3: Update θ_1 :

(a) Generate a candidate θ_1^* from the proposal distribution:

$$\theta_1^* \sim \mathcal{N}(\theta_1^{(j-1)}, \text{Var}(\theta_1^{(j-1)})), \text{ truncated at 0.}$$

(b) Compute the acceptance probability by making use of posterior marginal distribution given by (6):

$$r(\theta_1^{(j-1)}|\theta_1^*) = \min\left(1, \frac{\pi^*(\theta_1^*|\theta_2^{(j-1)}, \alpha^{(j-1)})}{\pi^*(\theta_1^{(j-1)}|\theta_2^{(j-1)}, \alpha^{(j-1)})}\right).$$

(c) Draw $u \sim U(0, 1)$. If $u \leq r(\theta_1^{(j-1)}|\theta_1^*)$, accept the candidate and set $\theta_1^{(j)} = \theta_1^*$. Otherwise, reject the candidate and set $\theta_1^{(j)} = \theta_1^{(j-1)}$.

Step 4: Update θ_2 :

(a) Generate a candidate θ_2^* from the proposal distribution:

$$\theta_2^* \sim \mathcal{N}(\theta_2^{(j-1)}, \text{Var}(\theta_2^{(j-1)})), \text{ truncated at 0.}$$

(b) Compute the acceptance probability by making use of posterior marginal distribution given by (7):

$$r(\theta_2^{(j-1)}|\theta_2^*) = \min\left(1, \frac{\pi^*(\theta_2^*|\theta_1^{(j-1)}, \alpha^{(j-1)})}{\pi^*(\theta_2^{(j-1)}|\theta_1^{(j-1)}, \alpha^{(j-1)})}\right).$$

(c) Draw $u \sim U(0, 1)$. If $u \leq r(\theta_2^{(j-1)}|\theta_2^*)$, accept the candidate and set $\theta_2^{(j)} = \theta_2^*$. Otherwise, reject the candidate and set $\theta_2^{(j)} = \theta_2^{(j-1)}$.

Step 5: Update α :

Generate $\alpha^* \sim \mathcal{N}(\alpha^{(j-1)}, \text{Var}(\alpha^{(j-1)}))$, truncated at 0, and likewise update $\alpha^{(j)}$ using the acceptance-rejection procedure by making use of the posterior marginal distribution given by (8).

Step 6: Increment the iteration index: Set $j = j + 1$.

Step 7: Repeat **Step 3** to **Step 6** until $j = N$. The result is the sequence of sampled parameter values:

$$((\theta_1^{(1)}, \theta_2^{(1)}, \alpha^{(1)}), \dots, (\theta_1^{(N)}, \theta_2^{(N)}, \alpha^{(N)})).$$

The aforementioned posterior marginal distributions do not correspond to well-known distributions. So we consider the following MH algorithm, Algorithm 1, to draw samples from these distributions

Algorithm 1 enables sampling from the posterior distribution using a normal distribution as the proposal distribution and an acceptance-rejection step. The normal distribution, $N(\cdot, \cdot)$, is commonly chosen for its simplicity, symmetry and flexibility. Its symmetric nature simplifies the calculation of the acceptance probability, while its tunable mean and variance facilitate effective exploration of the target distribution. However, since the normal distribution is defined over the entire real line and the parameters of the IER distribution are constrained to be positive, we use a truncated normal distribution, discarding any generated values less than 0 to ensure valid parameter updates. Using this algorithm, we generate N samples to compute the Bayesian estimate of β under the squared error loss function as follows:

$$\hat{\beta}_{SE} = \frac{1}{N - M} \sum_{j=M+1}^N \beta^{(j)}.$$

Here, M is defined as the burn-in period, referring to the initial set of iterations during which the Markov chain is allowed to converge to its stationary distribution. The samples generated during this period are typically discarded because they may not accurately represent the target distribution. The length of the burn-in period is not fixed and can vary depending on the specific problem and the convergence behavior of the chain. It is often determined through empirical analysis or convergence diagnostics, such as trace plots and autocorrelation functions. An appropriate burn-in period typically ranges from 10% to 25% of the total samples, depending on how quickly the chain is expected to converge to the target distribution. Details on considered length of the burn-in period, which supports the use of the MH algorithm, are thoroughly presented in [Section 6](#) on real data analysis, along with various graphical diagnostic tools.

The next objective is to compute Bayesian credible interval (BCI) estimates, which summarize the uncertainty about an unknown parameter based on its posterior distribution. Equal-tailed intervals may be preferred for symmetric distributions or when a straightforward interpretation is desired. To explore the behavior of the posterior distribution, it can be graphically plotted, which is particularly useful if equal-tailed intervals are of interest. Plotting the posterior can also help determine whether the distribution is unimodal. However, if there is no prior knowledge about the uni-modality or symmetry of the posterior – which may depend on the choice of parameter values – the highest posterior density (HPD) interval estimates are often preferred. HPD intervals are favored because they represent the shortest interval that contains the highest probability density. Consequently, if arrange the generated $N - M$ observations in ascending order as $(\theta_1^{(1)}, \theta_1^{(2)}, \theta_1^{(N-M)}), (\theta_2^{(1)}, \theta_2^{(2)}, \theta_2^{(N-M)})$ and $(\alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_1^{(N-M)})$, then the $100(1 - \rho)\%$ BCI estimate of $\beta = (\theta_1, \theta_2, \alpha)$ is given as $(\beta^{(\rho(N-M)/2)}, \beta^{((1-\rho/2)(N-M)})$. Further, the HPD interval estimate is an interval with the least interval length of all the BCI estimates.

Remark 1. We note that univariate priors are insufficient for modeling the statistical dependency between the shape and scale parameters. However, this dependency can be effectively captured using conditional prior distributions. The existing literature has explored the use of conditional priors for various lifetime model parameters. For instance, Singh and Tripathi (2015) discuss conditional priors in the context of the log-normal distribution. Next, we propose a conditional prior of the form: $\pi(\beta) = \pi(\alpha)\pi(\theta_1|\alpha)\pi(\theta_2|\alpha)$ where

$$\pi(\alpha|\mu_3, \nu_3) \propto \alpha^{\mu_3-1} e^{-\nu_3\alpha}, \mu_3 > 0, \nu_3 > 0,$$

$$\pi(\theta_i|\mu_i, \nu_i) \propto \alpha^{\mu_i} \theta_i^{\mu_i-1} e^{-\nu_i\theta_i\alpha}, i = 1, 2, \mu_i > 0, \nu_i > 0.$$

Here, μ_i and v_i ($i = 1, 2$), are the hyper-parameters responsible for reflecting the prior knowledge about the unknown parameters. Subsequently, the respective posterior marginal distributions turn out to have the following form

$$\begin{aligned}\pi^*(\theta_1|\theta_2, \alpha) &= \theta_1^{n_1+\mu_1-1} e^{-\theta_1 \sum_{i=1}^{n_1} t_i^{-2} - \theta_2 \sum_{i=1}^{n_2} S_i^{-2} - v_1 \alpha \theta_1} \prod_{i=1}^{n_1} (1 - e^{-\theta_1 t_i^{-2}})^{\alpha-1} \\ &\quad \times \prod_{i=1}^{n_2} S_i^{-3} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}, \\ \pi^*(\theta_2|\theta_1, \alpha) &= \theta_2^{n_2+\mu_2-1} e^{-\theta_2 \sum_{i=1}^{n_2} S_i^{-2} - v_2 \theta_2 \alpha} \prod_{i=1}^{n_2} S_i^{-3} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}, \\ \pi^*(\alpha|\theta_1, \theta_2) &= \alpha^{\mu_1+\mu_2+\mu_3+r-1} e^{-(\theta_1 v_1 + \theta_2 v_2 + \alpha v_3)} \prod_{i=1}^{n_1} (1 - e^{-\theta_1 t_i^{-2}})^{\alpha-1} \prod_{i=1}^{n_2} (1 - e^{-\theta_2 S_i^{-2}})^{\alpha-1} \\ &\quad \times (1 - e^{-\theta_2 S_T^{-2}})^{\alpha(n-r)}.\end{aligned}$$

Now, accordingly the MH algorithm can be implemented to compute the Bayesian estimates. We note that when the overestimation of a parameter may have more severe consequences than underestimation, or vice versa, asymmetrical loss functions or other types of loss functions, such as LINEX, Balanced and General Entropy, can be considered.

5. Simulation study and numerical example

This section provides a comprehensive exploration through simulations and numerical illustrations. The aim is to validate the proposed methods and demonstrate their applicability in practical scenarios. First, a simulation study is conducted to assess the performance and behavior of the estimates under different setups. Then, a numerical example is presented to further highlight its practical utility.

5.1. Simulation study

In this section, we perform a simulation study based on 1000 repetitions in the R-statistical programming language. We generate data from the IER distribution with parameters $a_0 = 0.7$, $a_1 = 1.5$, $\alpha = 1$. The stress levels are taken as $x_1 = 0.2$ and $x_2 = 1$, (i.e. $(\theta_1, \theta_2, \alpha) = (1, 2.2, 1)$), with step-stress changing time $\tau_1 = 0.6$ and 0.8 . In generation, we consider different sample sizes, such as $n = 30, 40$ and 50 , with the condition that r has 40% and 20% censoring and a prefixed termination time of $\tau_2 = 2$ for a hybrid censoring setup. For each sample, we compute the maximum likelihood estimates and elements of the variance-covariance matrix. We then repeat this process 1,000 times, and based on these repetitions, we compare the performance of the maximum likelihood estimates and the associated 95% asymptotic confidence interval estimates through absolute bias (ABias), mean squared errors (MSEs), average interval lengths (AILs) and coverage percentages (CPs). The computed values are reported in Table 1.

From the tabulated values, it can be seen that the higher value of n or a lower percentage of censoring r leads to better estimates both in terms of the ABias and MSE values. Also, the AILs tend to decrease with the improvement in CPs. We also observe that a higher value of step-stress-changing time in the experiment leads to an improvement in results. This is due to the fact that the greater the number of observed failures in the data, the better the estimates we can have, and subsequently, 95% CPs tend to improve with smaller AILs. One can also observe that the estimates for θ_2 generally have a higher ABias, MSE, and associated AILs as compared to θ_1 and α . A similar

Table 1. Abias, MSEs, AILs and CPs for MLEs when $(\theta_1, \theta_2, \alpha) = (1, 2.2, 1)$.

n	τ_1	$\hat{\beta}$	$r = 0.6 \times n$			$r = 0.8 \times n$		
			ABias	MSE	CP(AIL)	ABias	MSE	CP(AIL)
30	0.6	$\hat{\theta}_1$	0.7668	0.5880	92.7 (2.0963)	0.6407	0.4105	92.9 (1.9435)
		$\hat{\theta}_2$	0.7706	0.5938	92.1 (2.4583)	0.7332	0.5376	91.2 (2.3940)
		$\hat{\alpha}$	0.3881	0.1506	93.1 (1.9462)	0.3676	0.1351	93.3 (1.8496)
	0.8	$\hat{\theta}_1$	0.7053	0.4975	93.2 (1.9442)	0.6267	0.3928	92.4 (1.9426)
		$\hat{\theta}_2$	0.7566	0.5724	92.0 (2.4189)	0.6670	0.4449	93.9 (2.3336)
		$\hat{\alpha}$	0.3753	0.1408	92.6 (1.9094)	0.3599	0.1295	94.8 (1.7891)
40	0.6	$\hat{\theta}_1$	0.5969	0.3563	94.5 (1.9419)	0.5652	0.3194	94.4 (1.8185)
		$\hat{\theta}_2$	0.6378	0.4067	93.7 (2.3102)	0.6267	0.3928	93.8 (2.2125)
		$\hat{\alpha}$	0.3375	0.1139	93.8 (1.7285)	0.3212	0.1031	93.8 (1.6819)
	0.8	$\hat{\theta}_1$	0.5757	0.3315	94.9 (1.9024)	0.5393	0.2909	95.3 (1.8055)
		$\hat{\theta}_2$	0.6319	0.3992	93.0 (2.2538)	0.6059	0.3671	94.0 (2.2072)
		$\hat{\alpha}$	0.3242	0.1051	94.9 (1.7097)	0.3172	0.1006	92.3 (1.5683)
50	0.6	$\hat{\theta}_1$	0.5235	0.2740	95.8 (1.7358)	0.4906	0.2407	94.3 (1.6775)
		$\hat{\theta}_2$	0.5812	0.3378	93.9 (2.1523)	0.5528	0.3056	94.3 (2.0238)
		$\hat{\alpha}$	0.3104	0.0963	95.7 (1.5533)	0.2619	0.0686	94.7 (1.4573)
	0.8	$\hat{\theta}_1$	0.5012	0.2512	93.8 (1.6954)	0.4652	0.2164	95.4 (1.6270)
		$\hat{\theta}_2$	0.5583	0.3117	93.4 (2.1074)	0.5374	0.2888	94.8 (2.0065)
		$\hat{\alpha}$	0.2921	0.0853	95.9 (1.5478)	0.2390	0.0571	95.5 (1.4489)

Table 2. Abias, MSEs, AILs and CPs for MLEs when $(\theta_1, \theta_2, \alpha) = (1.39, 2.59, 1.5)$.

n	τ_1	$\hat{\beta}$	$r = 0.6 \times n$			$r = 0.8 \times n$		
			ABias	MSE	CP(AIL)	ABias	MSE	CP(AIL)
30	0.6	$\hat{\theta}_1$	0.6093	0.3712	90.3 (2.6216)	0.5780	0.3341	92.7 (2.3714)
		$\hat{\theta}_2$	0.7565	0.5723	91.8 (2.9481)	0.6182	0.3822	92.7 (2.8705)
		$\hat{\alpha}$	0.7077	0.5008	90.7 (1.7176)	0.6334	0.4012	92.4 (1.5212)
	0.8	$\hat{\theta}_1$	0.6074	0.3689	90.7 (2.4303)	0.5557	0.3088	91.5 (2.2924)
		$\hat{\theta}_2$	0.7364	0.5422	91.3 (2.8814)	0.5525	0.3053	90.2 (2.8267)
		$\hat{\alpha}$	0.6421	0.4123	93.4 (1.6708)	0.6103	0.3724	92.7 (1.4877)
40	0.6	$\hat{\theta}_1$	0.4168	0.1737	92.9 (2.2823)	0.3923	0.1539	91.3 (1.8666)
		$\hat{\theta}_2$	0.5084	0.2584	91.7 (2.4480)	0.4648	0.2161	92.2 (2.2848)
		$\hat{\alpha}$	0.5899	0.3479	90.7 (1.4595)	0.5694	0.3243	92.4 (1.4124)
	0.8	$\hat{\theta}_1$	0.4029	0.1623	90.8 (1.9785)	0.3808	0.1450	92.1 (1.7250)
		$\hat{\theta}_2$	0.4818	0.2321	91.8 (2.3233)	0.4640	0.2153	91.6 (2.2656)
		$\hat{\alpha}$	0.5797	0.3360	92.1 (1.4574)	0.5671	0.3217	92.9 (1.3971)
50	0.6	$\hat{\theta}_1$	0.3762	0.1416	91.8 (1.6926)	0.3477	0.1209	91.2 (1.6295)
		$\hat{\theta}_2$	0.3421	0.1171	92.9 (1.9497)	0.3256	0.1060	91.7 (1.7189)
		$\hat{\alpha}$	0.5508	0.3034	93.4 (1.2630)	0.5248	0.2754	92.9 (1.2420)
	0.8	$\hat{\theta}_1$	0.3661	0.1341	90.5 (1.6619)	0.3320	0.1102	92.3 (1.5161)
		$\hat{\theta}_2$	0.3401	0.1157	90.1 (1.8837)	0.2627	0.0690	92.2 (1.6731)
		$\hat{\alpha}$	0.5486	0.3010	93.7 (1.2537)	0.5175	0.2678	93.6 (1.1896)

type of findings are observed when we generate data from the IER distribution with $a_0 = 1.09, a_1 = 1.5, \alpha = 1.5$ (i.e. $(\theta_1, \theta_2, \alpha) = 1.39, 2.59$ and 1.5) with termination times such as $\tau_2 = 2$ corresponding to different step-stress changing times $\tau_1 = 0.6$ and 0.8 . The readings are reported in Table 2.

Next, under the Bayesian approach, we consider non-informative and informative priors. We denote the non-informative prior as prior 1, and the informative prior as prior 2. For prior 1, we take the values of hyper-parameters as $\mu_1 = v_1 = \mu_2 = v_2 = \mu_3 = v_3 = 0.0001$. However, for prior 2, we take hyper-parameter values as $\mu_1 = 2, v_1 = 2, \mu_2 = 4, v_2 = 2, \mu_3 = 2, v_3 = 2$ when

$(\theta_1, \theta_2, \alpha) = (1, 2.2, 1)$, and further hyper-parameter values are considered as $\mu_1 = 3, v_1 = 2, \mu_2 = 5, v_2 = 2, \mu_3 = 3, v_3 = 2$ when $(\theta_1, \theta_2, \alpha) = (1.39, 2.59, 1.5)$. Notice that the selection of hyper-parameter values under prior 2 is considered in such a way that the prior means always remain sufficiently close to the true values of parameters. For a discussion on the elicitation of hyper-parameter values from the past available data, one may also refer to Dey et al. (2016) and Singh and Tripathi (2018). The Bayesian estimates with ABias and MSEs and associated 95% HPD AILs and CPs are reported in Tables 3 and 4. The readings are based on the 1,000 considered samples after discarding the initial 200 samples as a burn-in period, which accounts for approximately 17% of the total samples. Furthermore, we used the ``boa'' package in R to calculate the HPD interval estimates. From reported readings, we notice that the estimates with increasing the value of n, r and τ_1 lead to the improvement of results based on ABias, MSEs, AILs, as well as CPs. In reality, the outcomes of the experiment are also influenced by the level of stress because failure can occur more quickly at higher stress levels. One can observe that with a smaller value of n and a higher value of r , the CPs are generally below the nominal level, but the situation improves with a higher

Table 3. Abias, MSEs, AILs and CPs for Bayesian estimates when $(\theta_1, \theta_2, \alpha) = (1, 2.2, 1)$.

n	r	τ_1	$\hat{\beta}$	Prior 1			Prior 2		
				ABias	MSE	CP(AIL)	ABias	MSE	CP(AIL)
30	18	0.6	$\hat{\theta}_1$	0.6365	0.4052	94.1 (1.9231)	0.5324	0.2834	94.9 (1.5967)
			$\hat{\theta}_2$	0.7494	0.5617	94.4 (2.1700)	0.6392	0.4086	95.9 (1.9927)
			$\hat{\alpha}$	0.2511	0.0630	92.8 (1.4232)	0.2251	0.0507	93.4 (1.3898)
		0.8	$\hat{\theta}_1$	0.6210	0.3856	94.2 (1.9052)	0.5110	0.2611	95.2 (1.5777)
			$\hat{\theta}_2$	0.7357	0.5413	93.3 (2.1614)	0.6281	0.3945	95.5 (1.9737)
			$\hat{\alpha}$	0.2503	0.0627	93.1 (1.4075)	0.2164	0.0468	94.3 (1.3782)
	24	0.6	$\hat{\theta}_1$	0.6032	0.3638	95.5 (1.8747)	0.4853	0.2355	95.8 (1.5604)
			$\hat{\theta}_2$	0.7214	0.5204	94.9 (2.1406)	0.6051	0.3662	94.7 (1.9678)
			$\hat{\alpha}$	0.2479	0.0614	92.9 (1.3882)	0.2008	0.0403	95.7 (1.3088)
		0.8	$\hat{\theta}_1$	0.6029	0.3634	95.4 (1.8217)	0.4682	0.2192	95.8 (1.5578)
			$\hat{\theta}_2$	0.5783	0.3344	94.8 (2.0065)	0.5450	0.2970	95.0 (1.9651)
			$\hat{\alpha}$	0.2350	0.0552	94.5 (1.3727)	0.1925	0.0370	95.1 (1.2877)
40	24	0.6	$\hat{\theta}_1$	0.5289	0.2798	94.1 (1.7741)	0.4575	0.2094	95.3 (1.5455)
			$\hat{\theta}_2$	0.5718	0.3270	93.1 (1.9611)	0.5283	0.2791	95.2 (1.9419)
			$\hat{\alpha}$	0.2267	0.0514	93.5 (1.3684)	0.1706	0.0291	95.7 (1.2607)
		0.8	$\hat{\theta}_1$	0.4901	0.2402	94.1 (1.7516)	0.4485	0.2012	94.7 (1.5356)
			$\hat{\theta}_2$	0.5564	0.3096	94.0 (1.9517)	0.5007	0.2507	94.6 (1.9112)
			$\hat{\alpha}$	0.2256	0.0509	94.2 (1.3502)	0.1559	0.0243	95.2 (1.2520)
	32	0.6	$\hat{\theta}_1$	0.4584	0.2102	95.9 (1.7376)	0.3649	0.1331	95.8 (1.5214)
			$\hat{\theta}_2$	0.5450	0.2970	94.8 (1.9477)	0.4947	0.2447	95.1 (1.9093)
			$\hat{\alpha}$	0.2173	0.0472	93.8 (1.2776)	0.1461	0.0213	95.7 (1.2361)
		0.8	$\hat{\theta}_1$	0.4446	0.1977	94.6 (1.6993)	0.3326	0.1107	95.4 (1.5018)
			$\hat{\theta}_2$	0.5330	0.2841	94.4 (1.9361)	0.4728	0.2236	95.6 (1.8681)
			$\hat{\alpha}$	0.2196	0.0482	93.7 (1.2690)	0.1311	0.0172	94.1 (1.2235)
	30	0.6	$\hat{\theta}_1$	0.4362	0.1902	94.5 (1.6675)	0.2839	0.0806	94.6 (1.4943)
			$\hat{\theta}_2$	0.4768	0.2274	93.6 (1.9293)	0.4354	0.1896	93.2 (1.8621)
			$\hat{\alpha}$	0.1972	0.0389	95.5 (1.2468)	0.1245	0.0155	95.9 (1.1845)
		0.8	$\hat{\theta}_1$	0.4045	0.1637	94.3 (1.6655)	0.2461	0.0606	95.3 (1.4659)
			$\hat{\theta}_2$	0.4572	0.2091	93.9 (1.8558)	0.4289	0.1840	94.5 (1.8136)
			$\hat{\alpha}$	0.1821	0.0332	94.4 (1.2257)	0.1181	0.0139	95.8 (1.1747)
	40	0.6	$\hat{\theta}_1$	0.3833	0.1469	94.2 (1.6279)	0.2284	0.0522	95.7 (1.4050)
			$\hat{\theta}_2$	0.4395	0.1932	94.9 (1.8291)	0.3981	0.1585	95.5 (1.7996)
			$\hat{\alpha}$	0.1768	0.0313	94.5 (1.2155)	0.1158	0.0134	95.2 (1.1519)
		0.8	$\hat{\theta}_1$	0.3666	0.1344	94.6 (1.6257)	0.1936	0.0375	95.8 (1.4021)
			$\hat{\theta}_2$	0.3964	0.1572	95.0 (1.8157)	0.3517	0.1237	94.7 (1.7838)
			$\hat{\alpha}$	0.1641	0.0269	95.3 (1.1820)	0.1071	0.0115	95.9 (1.1313)

Table 4. ABias, MSEs, AILs and CPs for Bayesian estimates when $(\theta_1, \theta_2, \alpha) = (1.39, 2.59, 1.5)$.

n	r	τ_1	$\hat{\beta}$	Prior 1			Prior 2		
				ABias	MSE	CP(AIL)	ABias	MSE	CP(AIL)
30	18	0.6	$\hat{\theta}_1$	0.4561	0.2081	93.4 (2.0025)	0.3325	0.1106	94.7 (1.8650)
			$\hat{\theta}_2$	0.7495	0.5618	92.8 (2.9163)	0.6935	0.4809	94.8 (2.7477)
			$\hat{\alpha}$	0.6543	0.4281	91.3 (1.6173)	0.5787	0.3349	93.9 (1.5173)
		0.8	$\hat{\theta}_1$	0.4437	0.1968	94.8 (1.9302)	0.3077	0.0947	94.5 (1.8548)
			$\hat{\theta}_2$	0.7129	0.5082	91.5 (2.7949)	0.5840	0.3411	93.1 (2.3232)
			$\hat{\alpha}$	0.6358	0.4043	90.4 (1.5481)	0.5695	0.3244	94.4 (1.4665)
	24	0.6	$\hat{\theta}_1$	0.4323	0.1869	94.1 (1.8662)	0.3176	0.1009	94.0 (1.7066)
			$\hat{\theta}_2$	0.5457	0.2978	92.7 (2.4877)	0.4200	0.1764	93.3 (2.2604)
			$\hat{\alpha}$	0.5669	0.3214	91.8 (1.4330)	0.5273	0.2781	92.4 (1.3988)
		0.8	$\hat{\theta}_1$	0.4133	0.1709	92.9 (1.8276)	0.2846	0.0810	95.4 (1.5947)
			$\hat{\theta}_2$	0.4774	0.2279	92.6 (2.3609)	0.3720	0.1384	93.7 (2.2204)
			$\hat{\alpha}$	0.5606	0.3143	90.9 (1.3892)	0.5522	0.3049	93.4 (1.2112)
40	24	0.6	$\hat{\theta}_1$	0.4103	0.1684	93.3 (1.6573)	0.2770	0.0767	96.8 (1.4685)
			$\hat{\theta}_2$	0.4606	0.2121	93.9 (2.2788)	0.3556	0.1264	94.1 (1.9827)
			$\hat{\alpha}$	0.5558	0.3089	90.2 (1.3812)	0.4378	0.1916	94.3 (1.2043)
		0.8	$\hat{\theta}_1$	0.3433	0.1178	94.2 (1.6425)	0.2652	0.0703	96.2 (1.3839)
			$\hat{\theta}_2$	0.3694	0.1365	93.5 (2.1628)	0.3576	0.1278	92.5 (1.9432)
			$\hat{\alpha}$	0.5351	0.2863	91.3 (1.3678)	0.3620	0.1310	94.5 (1.2169)
	32	0.6	$\hat{\theta}_1$	0.3332	0.1110	95.3 (1.5715)	0.2250	0.0506	95.6 (1.1834)
			$\hat{\theta}_2$	0.3421	0.1171	92.7 (2.0804)	0.3208	0.1029	93.8 (1.9400)
			$\hat{\alpha}$	0.5086	0.2587	92.3 (1.3629)	0.3333	0.1111	94.0 (1.2043)
		0.8	$\hat{\theta}_1$	0.3073	0.0944	94.3 (1.4993)	0.2033	0.0413	94.8 (1.1695)
			$\hat{\theta}_2$	0.2985	0.0891	92.5 (2.0398)	0.2757	0.0760	92.4 (1.8733)
			$\hat{\alpha}$	0.4567	0.2086	92.0 (1.3559)	0.3188	0.1016	93.6 (1.1896)
50	30	0.6	$\hat{\theta}_1$	0.2317	0.0537	94.6 (1.4660)	0.1969	0.0388	95.3 (1.1375)
			$\hat{\theta}_2$	0.2500	0.0625	93.7 (1.9151)	0.2360	0.0557	92.0 (1.7615)
			$\hat{\alpha}$	0.4401	0.1937	93.3 (1.2451)	0.3009	0.0905	92.5 (1.0789)
		0.8	$\hat{\theta}_1$	0.2295	0.0527	94.0 (1.3964)	0.1657	0.0275	96.0 (1.1337)
			$\hat{\theta}_2$	0.2238	0.0501	92.9 (1.7690)	0.2206	0.0487	94.7 (1.5836)
			$\hat{\alpha}$	0.3966	0.1573	92.4 (1.2008)	0.2864	0.0820	93.5 (0.9073)
	40	0.6	$\hat{\theta}_1$	0.2045	0.0418	95.1 (1.3743)	0.1534	0.0235	95.6 (1.1236)
			$\hat{\theta}_2$	0.2139	0.0458	93.0 (1.6731)	0.1979	0.0392	94.0 (1.4317)
			$\hat{\alpha}$	0.3233	0.1045	92.6 (1.2269)	0.2753	0.0758	95.2 (0.8835)
		0.8	$\hat{\theta}_1$	0.1986	0.0395	95.9 (1.3443)	0.1319	0.0174	95.4 (1.0411)
			$\hat{\theta}_2$	0.2079	0.0432	93.3 (1.5897)	0.1346	0.0181	94.8 (1.3592)
			$\hat{\alpha}$	0.2844	0.0809	93.5 (1.1895)	0.2383	0.0568	96.8 (0.7979)

number of units in the experiment and a smaller percentage of censoring. It is seen from the readings that the ABias and MSEs for α are comparatively smaller than those of the associated ABias and MSEs for θ_1 and θ_2 when $(\theta_1, \theta_2, \alpha) = (1, 2.2, 1)$. Further, the associated AILs are also smaller, but the associated CPs are lower than the nominal level as compared to estimates for θ_1 and θ_2 . However, this behaviour is not observed when $(\theta_1, \theta_2, \alpha) = (1.39, 2.59, 1.5)$. In fact, in this case, the ABias and MSEs for θ_1 are comparatively smaller, and furthermore, the associated AILs are smaller with high CPs as compared to the estimates for the θ_2 and α . The analysis reveals that among the priors considered, informative priors consistently outperform non-informative ones across all metrics. Notably, conditional priors also demonstrate superior performance compared to non-informative priors. This result aligns with theoretical expectations and is well-documented in the literature for various lifetime models. In conclusion, non-informative priors are preferable when there is little or no reliable prior information, as they minimize bias and allow the data to predominantly drive the inference. This approach produces more objective results, making it particularly useful in scientific studies where neutrality and transparency are important.

However, when relevant past data are available, hyper-parameter values can be used to incorporate prior knowledge. Additionally, when past data show a statistical dependency between the shape and scale parameters, conditional priors are recommended over independent gamma priors.

5.2. Numerical example

In the previous section, we presented estimates under different setups using a simulation study that incorporates randomness to capture inherent variability. This approach allows simulations to produce a variety of outcomes, including any of the cases described in Section 2, although case (iii) most frequently occurs due to our selected experimental setup. In this section, we illustrate case (i) and case (ii) using a generated data set. Generated data is often used to validate or test specific models. For instance, it can verify the accuracy of maximum likelihood or Bayesian estimates, serve as a benchmark for comparing experimental outcomes with theoretical expectations, and facilitate model calibration. Therefore, it plays a crucial role in validating assumptions, experimental designs, or analytical methods by providing a controlled data set for testing.

Consider an industrial company planning to conduct a life test experiment to assess the reliability of its components or making inference through point and interval estimates under step-stress conditions. Initially, the components will be exposed to a low-stress level of $x_1 = 0.2$ for a certain period, followed by exposure to a higher stress level of $x_2 = 1$ with the step-stress changing time set at $\tau_1 = 0.5$. Further, assume that the company has a total of $n = 30$ units available for testing, and due to budget and resource constraints, the company can either run the test for a maximum duration of $\tau_2 = 3$ time units or until a maximum of $m = 24$ units have failed, whichever occurs first. This setup allows the company to monitor the performance and reliability of the components under increasing stress levels while managing the risk of excessive failures or prolonged testing times through hybrid censoring. Here, the assumption of the CEM would be particularly useful for predicting the time to failure based on the cumulative damage accumulated during both stress phases. This helps in getting more accurate predictions and better inference about the reliability of the components under varying stress conditions.

Assume that the company is interested in making inference when the true parameter values are $a_0 = 1.7$, $a_1 = 1.5$ and $\alpha = 1.9$. Using the linear link function $\theta_i = a_0 + a_1 x_i$, we obtain $\theta_1 = 2.0$ and $\theta_2 = 3.2$. When generating the data set under this setup, we arrive at case (ii): when all the $r = m = 24$ failures occur after the stress-changing time $\tau_1 = 0.5$. In fact under this case, we get $r = m = 24$ with termination time $T = x_{24} = 2.7089$, and the following generated data set corresponding to the type-II censoring scheme: 0.6822, 0.8258, 0.9564, 1.0030, 1.0266, 1.1175, 1.3166, 1.3185, 1.3611, 1.3971, 1.4593, 1.4824, 1.5840, 1.6461, 1.7775, 1.9275, 2.0288, 2.1371, 2.1885, 2.2495, 2.2587, 2.5579, 2.5602, 2.7089. For this generated data set, we first compute the maximum likelihood estimates as $\hat{a}_0 = 2.0123$, $\hat{a}_1 = 1.4001$ and $\hat{\alpha} = 1.5412$. We further compute Bayesian estimates under Prior 1 with the hyper-parameters selected as $\mu_1 = 2$, $\nu_1 = 1$, $\mu_2 = 7$, $\nu_2 = 2$, $\mu_3 = 4$ and $\nu_3 = 2$ ensuring that the prior means remain close to the true parameter values.

Next, we present the estimates under the setup where the parameters are $a_0 = 0.8$, $a_1 = 1.2$ and $\alpha = 2$; the stress levels are $x_1 = 0.5$ and $x_2 = 0.7$. Therefore, from the linear link function relation $\theta_i = a_0 + a_1 x_i$, we obtain $\theta_1 = 1.4$ and $\theta_2 = 1.64$. Further, the step-stress changing time is set at $\tau_1 = 1.5$. Under the hybrid censoring setup, we consider $n = 30$, $m = 24$ and a termination time of $\tau_2 = 2$. This setup leads to case (i) with all the failures occurring before the stress changing time $\tau_1 = 1.5$ corresponding to type-II censoring. In generation, we get $r = 24$, $T = x_{24} = 1.3042$ and the following generated data set: 0.3990, 0.5951, 0.6046, 0.6321, 0.6788, 0.7810, 0.9423, 0.9587, 0.9591, 0.9756, 0.9770, 1.0052, 1.0141, 1.0493, 1.1006, 1.1059, 1.1822, 1.2252, 1.2264, 1.2276, 1.2377, 1.2766, 1.2854, 1.3042. For this generated data set, the maximum likelihood estimates are $\hat{a}_0 = 0.5677$, $\hat{a}_1 = 1.3996$ and $\hat{\alpha} = 1.9771$. Further, for Bayesian estimates under Prior 1, hyper-parameters are selected as $\mu_1 = 7$, $\nu_1 = 5$, $\mu_2 = 3$, $\nu_2 = 2$, $\mu_3 = 2$ and $\nu_3 = 1$. Table 5 presents the

Table 5. Point and interval estimates under case (i) and case (ii).

Case	$\hat{\beta}$	estimates		Interval estimates/Interval length	
		MLE	Bayes	CI/IL	HPD/IL
(ii)	$\hat{\theta}_1$	2.2923	2.0339	(0.2063, 4.0693)/3.8630	(0.6230, 2.3001)/1.6770
	$\hat{\theta}_2$	3.4125	3.2942	(1.2180, 5.6069)/4.3888	(1.7486, 4.7211)/2.9724
	$\hat{\alpha}$	1.5412	1.1799	(0, 3.4802)/3.4802	(0.3081, 1.5554)/1.2472
(i)	$\hat{\theta}_1$	1.2675	1.2316	(0.7110, 1.8240)/1.1130	(0.9358, 1.4679)/0.5321
	$\hat{\theta}_2$	1.5474	1.3102	(0, 3.3208)/3.3208	(0.0050, 2.8442)/2.8392
	$\hat{\alpha}$	1.9771	2.0342	(0.8819, 3.0724)/2.1904	(1.5391, 2.6203)/1.0811

maximum likelihood estimates, Bayesian estimates, and the associated interval estimates, along with their lengths. These estimates accommodate censoring schemes and incorporate prior knowledge, all of which contribute to a deeper understanding of component reliability. By applying these estimates, companies can make informed decisions, accurately predict component lifetimes and implement effective maintenance and design strategies to enhance overall product reliability. Finally, we mention that the relatively large bias observed for an estimate of a parameter in the numerical example can be attributed to the inherent variability in the specific dataset used. Since the data was randomly generated, certain datasets may naturally result in higher or lower biases in specific parameter estimates, even when the underlying model is correctly specified. To address this, our simulation study, conducted with 1000 repetitions, provides a broader evaluation of the estimation procedure's performance. The results demonstrate that, on average, the estimators are reliable and robust, underscoring the critical role of simulation studies in complementing numerical examples by offering a comprehensive perspective on the estimators' behavior.

6. Real data analysis

This section illustrates the proposed methods using two real data sets. The first data set, provided by McCool (1980), results from an Accelerated Life Testing experiment conducted with a 4-ball rolling contact test rig at Princeton Laboratories of Mobil Research and Development Company. This data set includes ordered failure times of 10 hardened steel specimens tested for rolling contact fatigue at four accelerated pressure levels (10^6 psi): 0.87, 0.99, 1.09 and 1.18. The normal pressure for failure times of steel specimens was 0.75. We focus on the two stress levels, $x_1 = 0.87$ psi and $x_2 = 0.99$ psi, which represent accelerated testing conditions designed to induce failures more quickly than under normal conditions. The failure time data for steel specimens is presented in Table 6 for reference purpose. It is important to note that the shape parameter governs the hazard rate behavior (e.g., increasing, decreasing or constant), and accurately capturing the shape of the distribution is critical for modeling failure mechanisms and understanding lifetime dynamics under accelerated conditions. In such scenarios, the Kolmogorov–Smirnov (KS) test statistic focuses on the alignment of the theoretical and empirical CDFs, making it particularly sensitive to the overall shape of the distribution. Therefore, we use the KS test statistic to assess the goodness-of-fit of the data set to the IER distribution. For the stress level, $x_1 = 0.87$ psi, the KS statistic is $D = 0.2003$ with

Table 6. Real data sets 1 and 2 under two stress levels.

Data set	stress level	Failure lifetimes						
1	$x_1 = 0.87$	1.67	2.20	2.51	3.00	3.90	4.70	7.53
		14.70	27.80	37.40				
	$x_2 = 0.99$	0.80	1.00	1.37	2.25	2.95	3.70	6.07
		6.65	7.05	7.37				
2	$x_1 = 0.2$	0.2733	0.2867	0.2933	0.3213			
	$x_2 = 1.0$	0.4387	0.4400	0.4433	0.4483	0.5117	0.5167	

a corresponding p -value of 0.7468. Similarly, for the stress level, $x_2 = 0.99$ psi, the KS statistic is $D = 0.2029$ with a corresponding p -value of 0.7337. These results suggest that the IER distribution is a reasonable model for fitting the real data set. Moreover, compared to commonly used lifetime models such as Weibull, lognormal and generalized half-normal distributions, the IER distribution yields a smaller KS distance, indicating a closer match in shape. This is particularly crucial for accelerated life tests, as deviations in the shape parameter can lead to incorrect interpretations of the hazard rate. We next simulate the data set using type-I hybrid censoring scenario, incorporating the last observation (37.40) into the observations at the 0.99 psi stress level. The simulation uses true parameter values of $a_0 = 0.5, a_1 = 0.8, \theta_1 = 1.19, \theta_2 = 1.29, \alpha = 2$ with $n = 20, \tau_1 = 38$ as stress changing time, $\tau_2 = 45$ as termination time, and censoring percentage as 30%. Hybrid censoring in this context enables a comprehensive analysis by accounting for both time and number of failures. This approach provides a more nuanced understanding of failure behavior and facilitates accurate modeling of the material's lifespan under varying stress levels.

Next, we calculate the point and interval estimates using the Newton-Raphson (NR) method. Due to the lack of prior information, the Bayesian estimates are computed using non-informative priors. All the results are reported in Table 7. The reported readings show that the point estimates from both MLE and Bayesian methods are close, indicating that the data and model are robust and that prior information (or the lack thereof) is not significantly influencing the results. Furthermore, the overlapping interval estimates from the classical and Bayesian approaches suggest that both methods provide similar information about the parameter's uncertainty, thereby increasing confidence in the robustness of the estimates. This statistical rigor provides reliability engineers with confidence in their models, leading to better decision-making when it comes to material selection, design modifications and stress-testing procedures.

A second real data set is taken from Greven et al. (2004), containing the failure times of 15 fish that initially swam for up to 90 min at a flow rate of 15 cm/sec. The failure time refers to when a fish experienced fatigue and changed its position. This data set includes four stress levels, each corresponding to an increase in the flow rate by 5 cm/sec every 20 min. The data set has also been analyzed by Nassar et al. (2021) and Dutta et al. (2023) for a simple step-stress case. Like previously data set, we similarly treat this data set as a simple step-stress data set by considering the first two stress levels, $x_1 = 0.2$ and $x_2 = 1$. For computational purposes, each data value has been adjusted by subtracting 50 and dividing by 150. The transformed data set is presented in Table 6.

The data set's structure makes it suitable for ALT and hybrid censoring analysis. By modeling the impact of stress on fatigue and failure, the study provides valuable insights into how accelerated conditions affect lifespan. Future studies can leverage these insights to estimate the lifespan of fish or similar subjects under varying conditions, contributing to better optimized experimental design and management practices. Next, we compute the KS test statistic for which the $D = 0.2805$, with a corresponding p -value of 0.8283, justifying the IER distribution for analysis of the data set when stress level is $x_1 = 0.2$. Likewise, for stress level $x_2 = 1.0$, the test statistic $D = 0.3651$ yielded a p -value 0.3191. To simulate type-I hybrid censored data from this data set, we use true parameter values of $a_0 = 0.5, a_1 = 0.8, \theta_1 = 0.66, \theta_2 = 1.30$ and $\alpha = 2$, with $n = 10, \tau_1 = 0.33$ as the stress change time,

Table 7. MLEs, Bayesian estimates and associated interval estimates.

Data set	$\hat{\beta}$	estimates		Interval estimates/Interval length	
		MLE	Bayes	CI/IL	HPD/IL
1	$\hat{\theta}_1$	17.4399	17.4268	(17.1772, 17.7026)/0.5254	(17.1705, 17.6988)/0.5282
	$\hat{\theta}_2$	12.9473	12.9450	(12.7193, 13.1752)/0.4559	(12.7153, 13.1757)/0.4603
	$\hat{a}$	0.8074	0.1259	(0.0047, 1.6100)/1.6053	(0.0622, 0.1904)/0.1282
2	$\hat{\theta}_1$	0.0981	0.2245	(0, 0.7347)/0.7347	(0.1365, 0.3366)/0.2001
	$\hat{\theta}_2$	0.9869	0.7799	(0.3071, 1.6668)/1.3596	(0.2497, 1.4151)/1.1653
	$\hat{a}$	2.7583	2.5913	(1.4989, 4.0177)/2.5188	(1.6444, 3.5836)/1.9391

$\tau_2 = 1$ as the termination time, and a censoring percentage of 20%. For illustration, we use $x_1 = 0.2$ and $x_2 = 1$ as stress levels. We then calculate the point and interval estimates using the NR method. Due to the lack of prior information, the Bayesian estimates are calculated using non-informative priors. The results are presented in Table 7. From reported readings, one can observe that estimates for the shape parameter α from MLE and Bayesian are close to each other as compared to the estimates for the scale parameters θ_1 and θ_2 . This shows that for α the prior information (or the lack thereof) is not significantly influencing the results as compared for the estimates for θ_1 and θ_2 .

Next, we discuss the validation of the drawn samples from the posterior distribution using the MH algorithm. Note that in MH algorithm implementation, the burn-in period allows the Markov chain to forget its starting point, which might be far from the target distribution, and stabilize within the region of the target distribution. In general, the length of the burn-in period is not fixed and can be determined through diagnostic methods. However, an appropriate burn-in period typically ranges from 10% to 25% of the total samples, depending on how quickly the chain is expected to converge to the target distribution. In this analysis, we considered 17% of the samples as the burn-in period. We use graphical diagnostic tools such as the autocorrelation function (ACF), trace plots and histograms with Gaussian kernel density estimates to assess the convergence of the stationary distributions for all parameters. Figures 1 and 2 illustrate these diagnostics for each

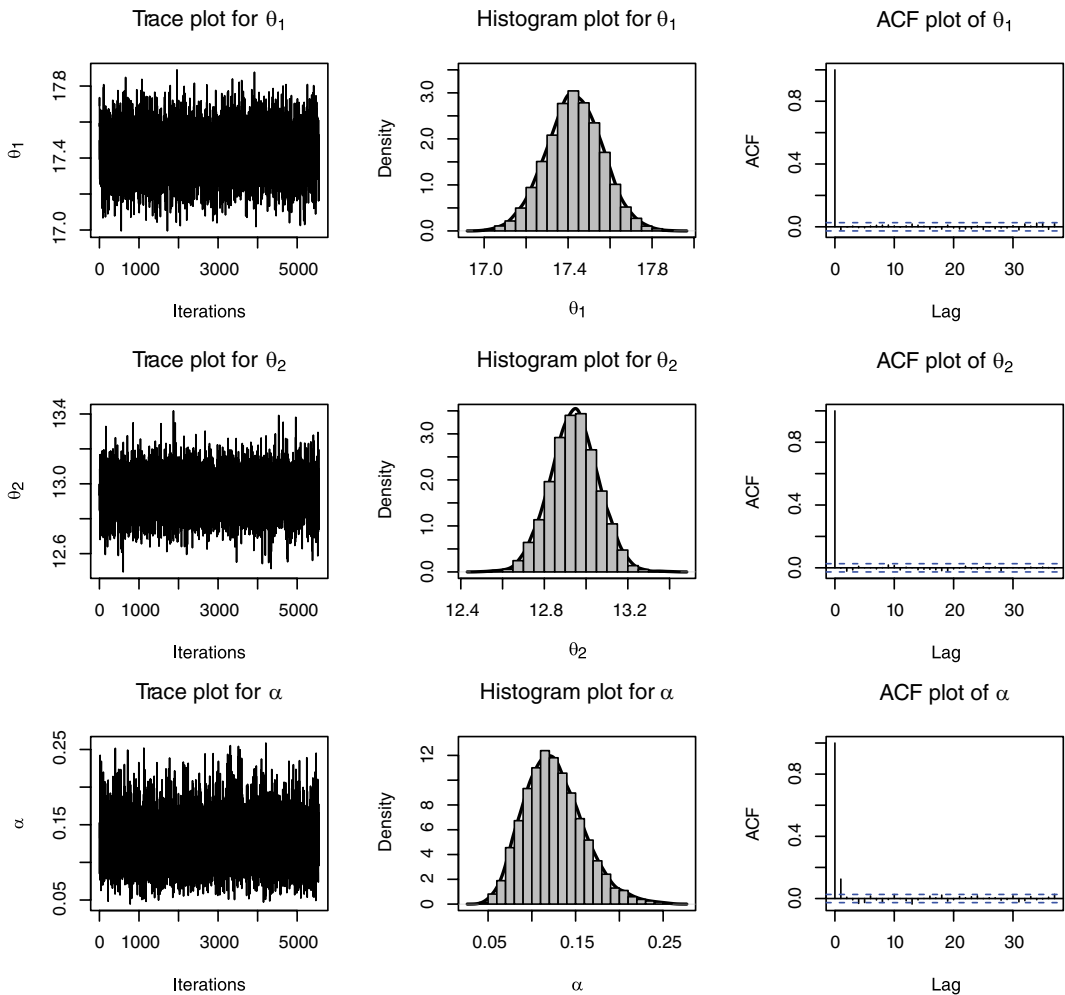


Figure 1. MCMC plot of real data set 1.

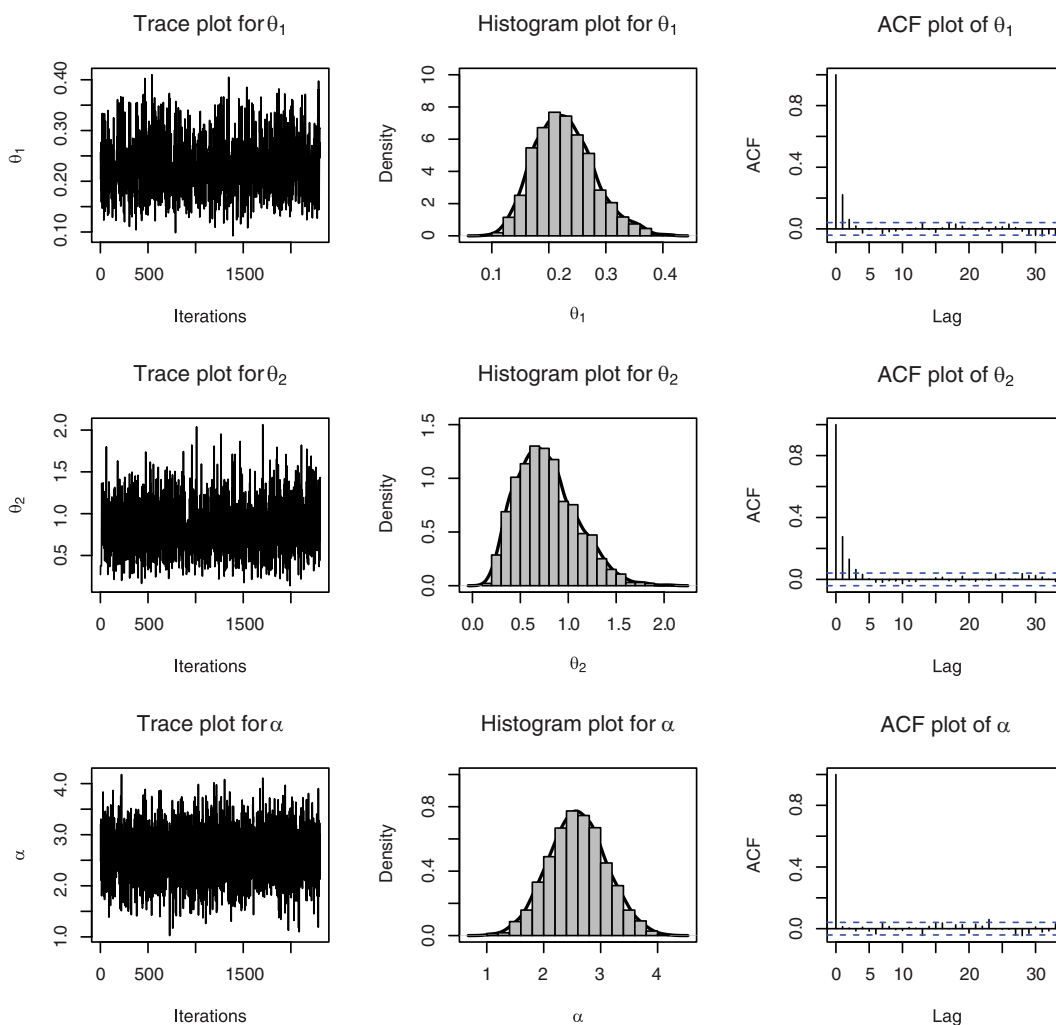


Figure 2. MCMC plot of real data set 2.

parameter. The trace plots show random dispersion around the mean value and effective mixing of the chains, while the ACF plots reveal very low autocorrelations, indicating good mixing. The histograms, displaying the marginal distributions of the parameters, are notably symmetrical, suggesting that the mean is the best estimate for the parameters. These graphical diagnostics demonstrate rapid MCMC convergence, thereby validating the use of the MH algorithm to compute Bayesian estimates.

7. Conclusion

In this paper, we analyzed lifetime data using the inverted exponentiated Rayleigh distribution within a simple step-stress accelerated life test framework, incorporating type-I hybrid censoring. We estimated the unknown parameters through maximum likelihood estimation, followed by the computation of asymptotic confidence intervals using the observed Fisher information matrix. For the Bayesian analysis, we considered both non-informative and informative priors, recognizing that the resulting posterior distribution was intractable. To address this, we employed the Metropolis-Hastings

algorithm to generate samples from the conditional posterior distributions, which were then used to compute Bayesian estimates under the squared error loss function. The validity of the MH algorithm was confirmed through graphical diagnostics, including autocorrelation functions, trace plots and histograms with Gaussian kernel density estimates, all of which demonstrated effective convergence to the stationary distributions for all parameters. Therefore, the Bayesian approach can be particularly useful in situations where historical data is scarce, allowing reliability practitioners to still obtain reliable estimates of model parameters and make data-driven decisions. In our simulation study, we observed the behavior of estimates under different setups and reported findings to analyze how factors such as sample size, termination time, permissible failures under hybrid censoring and the values of shape and scale parameters affect the estimates. These effects were assessed through their bias and mean squared errors. The findings offer reliability practitioners a method to simulate real-world scenarios, enabling them to predict failure times more accurately and optimize testing procedures. In the real data analysis, we observed the close agreement between point estimates from maximum likelihood and Bayesian methods indicating the robustness of the data and model, independent of prior information. The overlapping interval estimates from both approaches further reinforce the reliability of the parameter estimates, enhancing confidence in decision-making processes. Our analysis of a real data set, which examined the impact of stress on fatigue and failure, provides valuable insights into how accelerated conditions affect lifespan. These findings can guide future studies in estimating the lifespan of various subjects under different stress conditions, leading to more optimized experimental designs and improved management practices in the reliability engineering industry and business sectors. Finally, we mention that the development of optimal plans under classical and Bayesian approaches can be considered as a future work.

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No potential conflict of interest was reported by the author(s).

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References

- Asadi, S., & Panahi, H. (2022). Estimation of stress–strength reliability based on censored data and its evaluation for coating processes. *Quality Technology & Quantitative Management*, 19(3), 379–401. <https://doi.org/10.1080/16843703.2021.2001129>
- Balakrishnan, N., Cramer, E., & Kundu, D. (2023). *Hybrid censoring know-how: Designs and implementations*. Academic Press.
- Balakrishnan, N., & Xie, Q. (2007). Exact inference for a simple step-stress model with type-II hybrid censored data from the exponential distribution. *Journal of Statistical Planning and Inference*, 137(8), 2543–2563. <https://doi.org/10.1016/j.jspi.2006.04.017>
- Dey, S., Singh, S., Tripathi, Y. M., & Asgharzadeh, A. (2016). Estimation and prediction for a progressively censored generalized inverted exponential distribution. *Statistical Methodology*, 32, 185–202. <https://doi.org/10.1016/j.stamet.2016.05.007>
- Dutta, S., Sultana, F., & Kayal, S. (2023). Statistical inference for Gumbel type-II distribution under simple step-stress life test using type-II censoring. *Iranian Journal of Science*, 47(1), 155–173. <https://doi.org/10.1007/s40995-022-01394-3>
- Eapstein, B. (1954). Truncated life tests in the exponential case. *Annals of Mathematical Statistics*, 25(3), 555–564. <https://doi.org/10.1214/aoms/1177728723>
- Ghitany, M., Tuan, V., & Balakrishnan, N. (2014). Likelihood estimation for a general class of inverse exponentiated distributions based on complete and progressively censored data. *Journal of Statistical Computation and Simulation*, 84(1), 96–106. <https://doi.org/10.1080/00949655.2012.696117>
- Greven, S., John Bailer, A., Kupper, L. L., Muller, K. E., & Craft, J. L. (2004). A parametric model for studying organism fitness using step-stress experiments. *Biometrics*, 60(3), 793–799. <https://doi.org/10.1111/j.0006-341X.2004.00230.x>
- Kayal, T., Tripathi, Y. M., & Rastogi, M. K. (2018). Estimation and prediction for an inverted exponentiated Rayleigh distribution under hybrid censoring. *Communications in Statistics-Theory and Methods*, 47(7), 1615–1640. <https://doi.org/10.1080/03610926.2017.1322702>
- Kim, S.-H., & Sung, S.-I. (2023). Optimal design of cyclic-stress accelerated life tests for lognormal lifetime distribution. *Quality Technology & Quantitative Management*, 20(2), 236–255. <https://doi.org/10.1080/16843703.2022.2093577>
- Kotb, M. S., & El-Din, M. M. M. (2020). Guest editorial: Crisis management-from nuclear accidents to outbreaks of COVID-19 and infectious diseases. *IEEE Transactions on Reliability*, 69(3), 846–850. <https://doi.org/10.1109/TR.2020.3006408>
- Lin, C.-T., Chou, C.-C., & Balakrishnan, N. (2020). Planning step-stress test plans under type-I hybrid censoring for the log-location-scale distribution. *Statistical Methods & Applications*, 29(2), 265–288. <https://doi.org/10.1007/s10260-019-00476-8>
- Lindley, D. V. (1980). Approximate bayesian methods. *Trabajos de Estadística y de Investigación Operativa*, 31(1), 223–245. <https://doi.org/10.1007/BF02888353>
- Ma, J., Wang, L., Tripathi, Y. M., & Rastogi, M. K. (2023). Reliability inference for stress-strength model based on inverted exponential Rayleigh distribution under progressive type-II censored data. *Communications in Statistics-Simulation and Computation*, 52(6), 2388–2407. <https://doi.org/10.1080/03610918.2021.1908552>
- Maurya, R. K., Tripathi, Y. M., & Rastogi, M. K. (2019). Estimation and prediction for a progressively first failure censored inverted exponentiated Rayleigh distribution. *Journal of Statistical Theory and Practice*, 13(3), 1–48. <https://doi.org/10.1007/s42519-019-0038-7>
- McCool, J. I. (1980). Confidence limits for Weibull regression with censored data. *IEEE Transactions on Reliability*, R-29(2), 145–150. <https://doi.org/10.1109/TR.1980.5220758>
- Miller, R., & Nelson, W. (1983). Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, R-32(1), 59–65. <https://doi.org/10.1109/TR.1983.5221475>
- Nassar, M., Okasha, H., & Albassam, M. (2021). E-Bayesian estimation and associated properties of simple step–stress model for exponential distribution based on type-II censoring. *Quality and Reliability Engineering International*, 37(3), 997–1016. <https://doi.org/10.1002/qre.2778>
- Nelson, W. (1980). Accelerated life testing-step-stress models and data analyses. *IEEE Transactions on Reliability*, R-29(2), 103–108. <https://doi.org/10.1109/TR.1980.5220742>
- Nelson, W. B. (2009). *Accelerated testing: Statistical models, test plans, and data analysis*. John Wiley & Sons.

- Panahi, H., & Moradi, N. (2020). Estimation of the inverted exponentiated Rayleigh distribution based on adaptive type II progressive hybrid censored sample. *Journal of Computational and Applied Mathematics*, 364, 112345. <https://doi.org/10.1016/j.cam.2019.112345>
- Rastogi, M. K., & Tripathi, Y. M. (2014). Estimation for an inverted exponentiated Rayleigh distribution under type II progressive censoring. *Journal of Applied Statistics*, 41(11), 2375–2405. <https://doi.org/10.1080/02664763.2014.910500>
- Singh, S., & Tripathi, Y. M. (2015). Bayesian estimation and prediction for a hybrid censored lognormal distribution. *IEEE Transactions on Reliability*, 65(2), 782–795. <https://doi.org/10.1109/TR.2015.2494370>
- Singh, S., & Tripathi, Y. M. (2018). Estimating the parameters of an inverse Weibull distribution under progressive type-i interval censoring. *Statistical Papers*, 59(1), 21–56. <https://doi.org/10.1007/s00362-016-0750-2>
- Standard, M. (1977). 781-c, *reliability design qualification and production acceptance tests, exponential distribution*. Department of Defense.
- Tierney, L., & Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, 81(393), 82–86. <https://doi.org/10.1080/01621459.1986.10478240>
- Wang, L., Wu, S.-J., Tripathi, Y. M., & Dey, S. (2024). Estimation and prediction of modified progressive hybrid censored data from inverted exponentiated Rayleigh distribution. *Quality Technology & Quantitative Management*, 21(4), 502–524. <https://doi.org/10.1080/16843703.2023.2219556>
- Wu, W., Wang, B. X., Chen, J., Miao, J., & Guan, Q. (2023). Interval estimation of the two-parameter exponential constant stress accelerated life test model under type-II censoring. *Quality Technology & Quantitative Management*, 20(6), 751–762. <https://doi.org/10.1080/16843703.2022.2147688>
- Yadav, P., Kumar, D., & Singh, S. (2023). Statistical analysis for log-normal distribution in the presence of hybrid censoring and step-stress. *Strength of Materials*, 55(3), 552–565. <https://doi.org/10.1007/s11223-023-00548-x>

Appendix

$$\xi_1^{\theta_1\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_1(\theta_1, \theta_2)}{\partial \theta_1^2} = \frac{3}{2} \delta \theta_1^{-2} \theta_2 \sum_{i=1}^{n_2} S_i^{-3} (S_i^{-1} \delta - 1),$$

$$\xi_2^{\theta_1\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_2(\theta_1, \theta_2)}{\partial \theta_1^2} = -\frac{1}{4} \theta_1^{-2} \delta \sum_{i=1}^{n_2} S_i^{-1} (S_i^{-1} \delta - 3),$$

$$\xi_3^{\theta_1\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_3(\theta_1, \theta_2)}{\partial \theta_1^2} = \theta_2 (A_{2i} (\frac{3}{2} \delta \theta_1^{-2} \sum_{i=1}^{n_2} S_i^{-3} (S_i^{-1} \delta - 1)) - S_i^{-6} \delta^2 \theta_1^{-2} \theta_2 e^{-\theta_2 S_i^{-2}} (A_{2i} + 1)),$$

$$\xi_4^{\theta_1\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_4(\theta_1, \theta_2)}{\partial \theta_1^2} = \theta_2 (A_{2T} (\frac{3}{2} \delta \theta_1^{-2} \sum_{i=1}^{n_2} S_T^{-3} (S_T^{-1} \delta - 1)) - S_T^{-6} \delta^2 \theta_1^{-2} \theta_2 e^{-\theta_2 S_T^{-2}} (A_{2T} + 1)),$$

$$\xi_1^{\theta_2\theta_2}(\theta_1, \theta_2) = \frac{\partial^2 \xi_1(\theta_1, \theta_2)}{\partial \theta_2^2} = -B \sum_{i=1}^{n_2} S_i^{-3} (3S_i^{-1} \delta + 1),$$

$$\xi_2^{\theta_2\theta_2}(\theta_1, \theta_2) = \frac{\partial^2 \xi_2(\theta_1, \theta_2)}{\partial \theta_2^2} = -\frac{\tau_1}{4\theta_1} \sum_{i=1}^{n_2} (S_i^{-1} \theta_1^{-1} \left(\frac{\theta_2}{\theta_1}\right)^{-1} (\tau_1 - \left(\frac{\theta_2}{\theta_1}\right)^{-\frac{1}{2}})),$$

$$\xi_3^{\theta_2\theta_2}(\theta_1, \theta_2) = \frac{\partial^2 \xi_3(\theta_1, \theta_2)}{\partial \theta_2^2} = -\sum_{i=1}^{n_2} A_{2i} (A_{2i} (S_i^{-3} \delta + S_i^{-2})^2 + (S_i^{-3} \delta + S_i^{-2})^2 - BS_i^{-3} (-3S_i^{-1} \delta - 1)),$$

$$\xi_4^{\theta_2\theta_2}(\theta_1, \theta_2) = \frac{\partial^2 \xi_4(\theta_1, \theta_2)}{\partial \theta_2^2} = -A_{2T} (A_{2T} (S_T^{-3} \delta + S_T^{-2})^2 + (S_T^{-3} \delta + S_T^{-2})^2 - BS_T^{-3} (-3S_T^{-1} \delta - 1)),$$

$$\xi_1^{\theta_2\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_1(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = \theta_1^{-1} \sum_{i=1}^{n_2} S_i^{-3} (-3\delta S_i^{-1} B\theta_2 + B\theta_2 + \delta),$$

$$\xi_2^{\theta_2\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_2(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = \frac{1}{2} \theta_1^{-1} BS_i^{-1} (S_i^{-1} \delta - 1),$$

$$\xi_3^{\theta_2\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_3(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = -A_{2i} S_i^{-3} \delta (A_{2i} (S_i^{-3} \delta + S_i^{-2}) \theta_2 + (S_i^{-3} \delta + S_i^{-2}) \theta_2 + 3B\theta_2 S_i^{-1} - B\theta_2 - 1),$$

$$\xi_4^{\theta_2\theta_1}(\theta_1, \theta_2) = \frac{\partial^2 \xi_4(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = -A_{2T} S_T^{-3} \delta (A_{2T} (S_T^{-3} \delta + S_T^{-2}) \theta_2 + (S_T^{-3} \delta + S_T^{-2}) \theta_2 + 3B\theta_2 S_T^{-1} - B\theta_2 - 1).$$