

Inverse unit exponential probability distribution: Classical and Bayesian inference with applications

Cite as: AIP Advances 14, 055108 (2024); doi: 10.1063/5.0210828

Submitted: 27 March 2024 • Accepted: 22 April 2024 •

Published Online: 7 May 2024



Najwan Alsadat,^{1,a)} Caner Tanış,^{2,b)} Laxmi Prasad Sapkota,^{3,c)} Anoop Kumar,^{4,d)} Waleed Marzouk,^{5,e)} and Ahmed M. Gemeay^{6,f)}

AFFILIATIONS

¹ Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia

² Department of Statistics, Çankırı Karatekin University, Çankırı, Türkiye

³ Department of Statistics, Tribhuvan University, Tribhuvan Multiple Campus, Palpa, Nepal

⁴ Department of Statistics, Faculty of Basic Science, Central University of Haryana, Mahendergarh 123031, India

⁵ Faculty of Graduate Studies for Statistical Research, Department of Mathematical Statistics, Cairo University, Giza 12613, Egypt

⁶ Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

^{a)} nalsadat@ksu.edu.sa

^{b)} canertanis@karatekin.edu.tr

^{c)} Author to whom correspondence should be addressed: laxmisapkota75@gmail.com

^{d)} anoopk@cuh.ac.in

^{e)} wgm-74@hotmail.com

^{f)} ahmed.gemeay@science.tanta.edu.eg

ABSTRACT

This article examines the new inverse unit exponential distribution, utilizing both classical and Bayesian methodologies; it begins by presenting the general properties of the proposed model, highlighting characteristic features, such as the presence of a reverse-J or increasing and inverted bathtub-shaped hazard rate function. Furthermore, it explores various statistical properties of the suggested distribution. It employs 12 methods to estimate the associated parameters. A Monte Carlo simulation is conducted to evaluate the accuracy of the estimation procedure. Even for small samples, the results indicate that biases and mean square errors decrease as the sample size increases, demonstrating the robustness of the estimation method. The application of the suggested distribution to real datasets is then discussed. Empirical evidence, using model selection criteria and goodness-of-fit test statistics, supports the assertion that the suggested model outperforms several existing models considered in the study. This article extends its analysis to the Bayesian framework. Using the Hamiltonian Monte Carlo with the no U-turn sampling algorithm, 8000 real samples are generated. The convergence assessment reveals that the chains are convergent and the samples are independent. Subsequently, using the posterior samples, the parameters of the proposed model are estimated, and credible intervals and highest posterior density intervals are computed to quantify uncertainty. The applicability of the suggested model to real data under both classical and Bayesian methodologies provides insights into its statistical properties and performance compared to existing models.

© 2024 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/5.0210828>

I. INTRODUCTION

Many statistical distributions have recently been proposed in the literature as alternatives to current distributions. Generating distributions aims to provide the literature with a more flexible distribution than its competitors in modeling real-world data and statistical properties. In this regard, many authors aim to suggest

a new distribution to provide novelty in the literature. This study introduces a new distribution called inverse unit exponential distribution and its distributional properties. Some of the several papers related to this study are listed below.

Barco *et al.*¹ considered a new inverse power Lindley distribution as an alternative to Lindley distribution. Then, Jan *et al.*² generalized the inverse power Lindley distribution generated by

Barco *et al.*¹ and suggested the exponentiated inverse power Lindley distribution as an alternative to Lindley, power Lindley, and the modified ones. Hassan and Amal³ proposed the inverse power Lomax distribution. The main properties of the suggested distribution are implemented, and the superiority of the suggested distribution is illustrated via real-life data examples in Ref. 3. Sobhi and Mashail⁴ introduced the inverse power logistic-exponential distribution. The authors showed that in real-world data applications, the proposed distribution is handy in the modeling of real-life data and is a good alternative to competitors. Chesneau and Agiwal⁵ introduced the inverse power Muth distribution. In addition, the classical Bayesian methods are implemented to estimate the parameters of the proposed distribution. Akzzaz and Abd El-Monsef⁶ described a new distribution, called the inverse power Maxwell distribution.⁶ They provided several distributional properties and real-world data examples. Yildirim *et al.*⁷ introduced the power unit Burr-XII distribution. They investigated six different point estimation methods and examined the proposed distribution's distributional properties. Alsadat *et al.*⁸ suggested the inverse unit Teissier distribution. They focused on ten different estimators to estimate the unknown parameters of the introduced distribution. Alsaahangiti *et al.*⁹ proposed the inverse power generalized Maxwell distribution, whereas Sapkota and Kumar¹⁰ studied inverse transformation of the power Cauchy distribution. In this study, we also have used the inverse transformation technique to develop a new generalized model. We analyzed this model using both classical and Bayesian approaches.

In the past, conducting Bayesian analysis on models with numerous parameters posed significant challenges due to the intricate nature of the posterior distribution. However, with the rapid advancement of programming languages and computational tools, such as JAGS, OpenBUGS, and Stan for Bayesian analysis, researchers have been increasingly drawn toward Bayesian inference.

This paper explores the utilization of Bayesian methodology to fit the inverse unit exponential model for lifetime data using Stan. The Bayesian approach is facilitated through the *pystan* package in Python and the *rstan* package in R (see Ref. 11), providing an ideal environment for Bayesian modeling. Stan, a programming language tailored for statistical modeling, particularly stands out in Bayesian estimation tasks, offering quicker and more efficient computations compared to alternatives such as JAGS/BUGS (Carpenter *et al.*, 2017).¹² Moreover, simulation is a viable technique, especially when direct sampling from the posterior distribution $p(\xi|y)$ is impractical. Simulation, particularly based on Markov chain Monte Carlo (MCMC) methods, proves invaluable for obtaining reliable results across a range of complex problems.¹³

Gibbs sampling, the Hamiltonian Monte Carlo (HMC) algorithm, and the Metropolis-Hastings algorithm are prominent MCMC techniques that make computationally challenging tasks tractable. Previous studies, such as those by Abujarad and Khan (2018)¹⁴ and Sapkota (2022),¹⁵ have utilized similar techniques to analyze survival models. Moreover, Sapkota *et al.* (2023)¹⁶ have used Bayesian inferences to examine the arctan exponential model.

This paper aims to accomplish the following objectives through Bayesian analysis:

- To show that defining a Bayesian model entails specifying the likelihood and prior distribution.

- To implement an R code for approximating posterior densities using Stan.
- To present numeric and graphical summaries of the posterior densities for comprehensive interpretation.

This paper is organized as follows: in Sec. II, we formulate the new distribution. Section III examines some properties of the new model, such as quantile function and linear representation. In Sec. IV, we investigate 12 different methods to estimate the parameters of our model, namely, maximum likelihood, Anderson-Darling, Cramér-von Mises, maximum product spacings, least squares, right tail Anderson-Darling, weighted least squares, left tail Anderson-Darling, minimum spacing absolute distance, minimum spacing absolute log distance, Anderson-Darling method 2LT left-tail second order, and Kolmogorov methods. We considered an extensive Monte Carlo simulation study to evaluate the performances of the emphasized estimators in terms of mean squared error criterion in Sec. V. Then, we present two real-world data examples (flood charges and vehicle pass datasets) to illustrate the usefulness of our model in modeling real-life data. Section VII provides a Bayesian analysis of our model. Finally, the conclusions are given in Sec. VIII.

II. MODEL FORMULATION

Bakouch *et al.*¹⁷ introduced a unit distribution called unit exponential probability distribution (UEPrD) and defined their cumulative distribution function (CDF) and probability distribution function (PDF), respectively, as follows:

$$B(x) = 1 - e^{\tau \left(1 - \left(\frac{x+1}{1-x}\right)^\mu\right)}, \quad 0 < x < 1, \quad \tau, \mu > 0, \quad (1)$$

$$b(x) = \frac{2\tau\mu \left(\frac{x+1}{1-x}\right)^\mu e^{\tau \left(\frac{x+1}{1-x}\right)^\mu}}{1 - x^2}. \quad (2)$$

By applying the inverse transformation on the CDF (1), we have a new distribution called inverse UEPrD (IUEPrD). The CDF and PDF of the new model are given by

$$G(x) = e^{\tau \left(\frac{x+1}{x-1}\right)^\mu}, \quad x > 1, \quad \tau, \mu > 0, \quad (3)$$

$$g(x) = \frac{2\mu\tau \left(\frac{x+1}{x-1}\right)^\mu e^{\tau \left(\frac{x+1}{x-1}\right)^\mu}}{x^2 - 1}. \quad (4)$$

A. Survival function

The survival function of the IUEPrD can be expressed as

$$S(x) = 1 - e^{\tau \left(\frac{x+1}{x-1}\right)^\mu}, \quad x > 1, \quad \tau, \mu > 0. \quad (5)$$

B. Hazard rate function

The hazard rate function (HRF) of the IUEPrD is the ratio of PDF to survival function, and it can be presented as

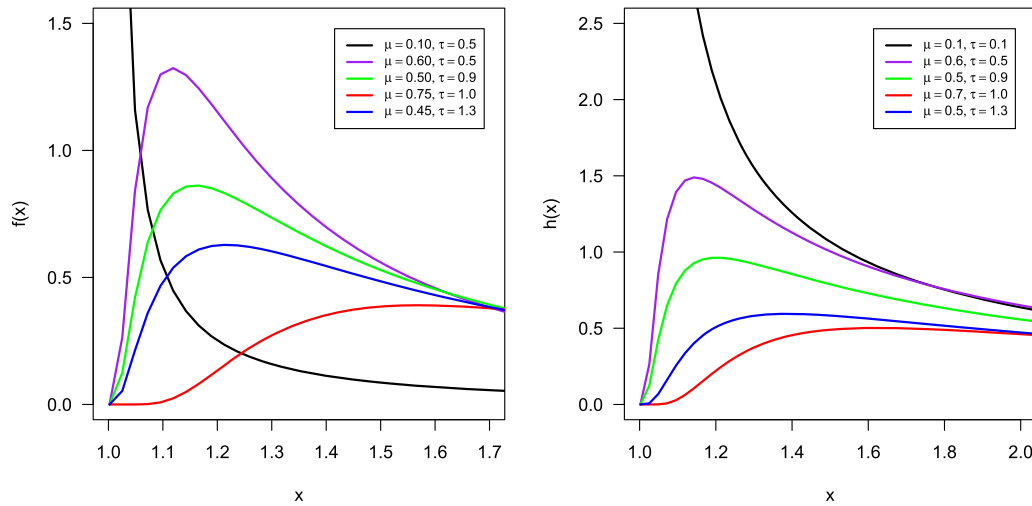


FIG. 1. Various shapes of PDF and HRF of IUEPrD.

$$h(x) = \frac{2\mu\tau\left(\frac{x+1}{x-1}\right)^\mu e^{\tau-\tau\left(\frac{x+1}{x-1}\right)^\mu}}{(x^2-1)\left[1 - e^{\tau-\tau\left(\frac{x+1}{x-1}\right)^\mu}\right]}. \quad (6)$$

A graphical representation of the PDF and HRF of the suggested model is presented in Fig. 1 using different values for parameters μ and τ .

III. STATISTICAL PROPERTIES

A. Quantile function

The quantile function of IUEPrD, denoted by QF, is a fundamental concept in statistics and probability theory. It plays a crucial role in describing the distribution of a random variable and is particularly useful in analyzing and summarizing data. QF can be expressed as

$$QF = \frac{\left(\frac{\tau - \log(p)}{\tau}\right)^{1/\mu} + 1}{\left(\frac{\tau - \log(p)}{\tau}\right)^{1/\mu} - 1}, \quad 0 < p < 1. \quad (7)$$

B. Linear representation

A linear representation of the CDF of our proposed model in Eq. (3) is derived to simplify our calculations as follows:

$$G(x) = e^{\tau - \tau\left(\frac{x+1}{x-1}\right)^\mu}.$$

After applying exponential expansion to the previous equation, we have

$$G(x) = e^\tau \sum_{h=0}^{\infty} \frac{(-1)^h \tau^h \left(\frac{1}{x} + 1\right)^{h\mu} \left(1 - \frac{1}{x}\right)^{h(-\mu)}}{h!}.$$

Again, applying the binomial expansion and doing some simplifications, we have

$$G(x) = e^\tau \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{\mu h}{i} \binom{k+h\mu-1}{\mu h-1} \frac{(-1)^h \tau^h}{h!} x^{-i-k}. \quad (8)$$

Then, a linear representation of the PDF of our model is

$$\begin{aligned} g(x) &= e^\tau \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{\mu h}{i} \binom{k+h\mu-1}{\mu h-1} \\ &\quad \times \frac{(-1)^{h+1} \tau^h (i+k)}{h!} x^{-i-k-1} \\ &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} x^{-i-k-1}, \end{aligned} \quad (9)$$

$$\text{where } \rho_{ihk} = e^\tau \binom{\mu h}{i} \binom{k+h\mu-1}{\mu h-1} \frac{(-1)^{h+1} \tau^h (i+k)}{h!}.$$

C. Moments

Moments are important statistical measures; here, we express the r th raw moment taken from the origin as

$$\begin{aligned} \mu'_r &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \int_1^{\infty} x^{-(i+k+1-r)} dx \\ &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-r}, \end{aligned} \quad (10)$$

where $i+k \neq r$. Using Eq. (10), we can compute the first four raw moments μ'_1 (mean), μ'_2 , μ'_3 , and μ'_4 as follows:

$$\begin{aligned} \mu'_1 &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-1}, \\ \mu'_2 &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-2}, \end{aligned}$$

$$\mu'_3 = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-3},$$

$$\mu'_4 = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-4}.$$

Hence, the variance $V(X)$ is

$$V(x) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-2} - \left\{ \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{1}{i+k-1} \right\}^2.$$

D. Central moments

The n th central moment of the IUEPrD can be obtained by using the mathematical relation

$$\begin{aligned} \mu_n &= \sum_{j=0}^n (-1)^j \binom{n}{j} (\mu'_1)^j \mu'_{n-j} \\ &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} (\mu'_1)^j \rho_{ihk} \frac{1}{i+k-n+j}. \end{aligned}$$

E. Moment generating function (MGF)

The MGF can also be used to obtain various orders of the moments, and it can be expressed as

$$\begin{aligned} M_X(m) &= \sum_{p=0}^{\infty} \frac{m^p}{p!} \mu'_p \\ &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{p=0}^{\infty} \frac{m^p \rho_{ihk}}{p!} \frac{1}{i+k-r}. \end{aligned}$$

F. Mean deviation (MD) from mean

The MD about mean is a statistical measure used to measure the variability in the dataset. We use the following mathematical expression to compute it for the IUEPrD:

$$\begin{aligned} MD(\mu) &= \int_0^{\infty} |x - \mu| f(x) dx \\ &= 2\mu F(\mu) - 2\mu + \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{\mu^{1-i-k}}{1-i-k}, \end{aligned}$$

where $F(\cdot)$ is the CDF defined in (3) and μ is the mean of IUEPrD.

G. Mean residual life function

The mean residual life function can be given as

$$\begin{aligned} MRL(x) &= \frac{1}{S(x)} \int_x^{\infty} x f(x) dx - x \\ &= \frac{1}{S(x)} \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \rho_{ihk} \frac{x^{1-i-k}}{1-i-k} - x, \end{aligned}$$

where $S(x)$ is the survival function defined in (5).

IV. ESTIMATION METHODS

To estimate the parameters of the proposed model, we have used several common estimation methods, which are presented in the following.

A. Method of maximum likelihood (EM_1)

This method is used to estimate the parameters of the IUEPrD by optimizing the log-likelihood function, which can be expressed as

$$L(\mu, \tau \mid x_1, x_2, \dots, x_n) = \prod_{i=1}^n g(x_i).$$

Then, the log-likelihood function is

$$\ell(\mu, \tau \mid x_1, x_2, \dots, x_n) = \ln L(\mu, \tau \mid x_1, x_2, \dots, x_n) = \sum_{i=1}^n \ln g(x_i).$$

Thus, the log-likelihood function for the given distribution is

$$\begin{aligned} \ell(\mu, \tau \mid x_1, x_2, \dots, x_n) &= \sum_{i=1}^n \left[\ln(2\mu\tau) + \mu \ln \left(\frac{x_i + 1}{x_i - 1} \right) \right. \\ &\quad \left. + \tau - \tau \left(\frac{x_i + 1}{x_i - 1} \right)^{\mu} - \ln(x_i^2 - 1) \right]. \end{aligned}$$

B. Method of Anderson-Darling (EM_2)

Suppose that we have an ordered random sample of size n given by $X_1, \dots, X_n$, where each X_i is distributed according to the IUEPrD with parameters μ and τ . To estimate the IUEPrD coefficients using the Anderson-Darling estimation approach, we have to minimize the following equation:

$$A(x_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{n-i+1:n})],$$

where $F(\cdot)$ and $S(\cdot)$ are the CDF and survival functions of the IUEPrD.

C. Method of Cramér-von Mises (EM_3)

To estimate the IUEPrD parameters using the Cramér-von Mises estimation approach, we have to minimize the following equation:

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2.$$

D. Method of maximum product of spacings (MPS) (EM_4)

The maximum product of spacings (MPS) is a nonparametric estimation technique commonly employed in statistics to estimate the parameters of a probability model. The methodology of MPS revolves around maximizing the product of the observed spacings between ordered data points. It finds significant application in estimating parameters for continuous probability distributions,

$$\delta(x_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \omega_i(x_i),$$

where $\omega_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$, $F(x_{0:n}) = 0$, and $F(x_{n+1:n}) = 1$.

TABLE I. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 0.9$ and $\mu = 0.3$.

n	Est.	Est. Par.	EM ₁	EM ₂	EM ₃	EM ₄	EM ₅	EM ₆	EM ₇	EM ₈	EM ₉	EM ₁₀	EM ₁₁	EM ₁₂
25	BIAS	$\hat{\tau}$	0.34827 ^[1]	0.35961 ^[9]	0.36337 ^[10]	0.34892 ^[2]	0.35897 ^[7]	0.36628 ^[11]	0.35072 ^[3]	0.35093 ^[4]	0.35954 ^[8]	0.35616 ^[6]	0.36749 ^[12]	0.35491 ^[5]
		$\hat{\mu}$	0.09723 ^[8]	0.09849 ^[9]	0.10316 ^[10]	0.08509 ^[1]	0.09597 ^[7]	0.11234 ^[12]	0.09169 ^[4]	0.09247 ^[5]	0.09119 ^[3]	0.08718 ^[2]	0.09309 ^[6]	0.10894 ^[11]
	MSE	$\hat{\tau}$	0.16418 ^[1]	0.17271 ^[6]	0.17634 ^[10]	0.16958 ^[5]	0.17416 ^[7]	0.17559 ^[8]	0.16657 ^[2]	0.16865 ^[4]	0.17772 ^[11]	0.17616 ^[9]	0.1823 ^[12]	0.16786 ^[3]
		$\hat{\mu}$	0.01503 ^[8]	0.01516 ^[9]	0.01631 ^[10]	0.01162 ^[1]	0.01449 ^[7]	0.01904 ^[12]	0.01322 ^[4]	0.01417 ^[6]	0.01321 ^[3]	0.01238 ^[2]	0.01395 ^[5]	0.018 ^[11]
	MRE	$\hat{\tau}$	0.38697 ^[1]	0.39957 ^[8]	0.40374 ^[9]	0.38769 ^[2]	0.39885 ^[6]	0.40698 ^[10]	0.38969 ^[3]	0.38992 ^[4]	0.39948 ^[7]	0.39573 ^[5]	0.40833 ^[11.5]	0.40833 ^[11.5]
$\sum Ranks$			26 ^[3.5]	49 ^[8]	58 ^[10]	12 ^[1]	40 ^[7]	63 ^[12]	20 ^[2]	28 ^[5]	35 ^[6]	26 ^[3.5]	58.5 ^[11]	52.5 ^[9]
90	BIAS	$\hat{\tau}$	0.26055 ^[1]	0.27883 ^[4]	0.28753 ^[7]	0.27829 ^[3]	0.29502 ^[8]	0.32143 ^[12]	0.2859 ^[6]	0.27506 ^[2]	0.28239 ^[5]	0.29995 ^[10]	0.29534 ^[9]	0.30261 ^[11]
		$\hat{\mu}$	0.05689 ^[2]	0.0598 ^[5]	0.07161 ^[10]	0.0517 ^[1]	0.06982 ^[9]	0.08415 ^[12]	0.06376 ^[8]	0.05704 ^[4]	0.0603 ^[6]	0.05702 ^[3]	0.06052 ^[7]	0.07693 ^[11]
	MSE	$\hat{\tau}$	0.10244 ^[1]	0.11788 ^[3]	0.12033 ^[4]	0.12382 ^[7]	0.12894 ^[8]	0.14551 ^[12]	0.12116 ^[5]	0.11427 ^[2]	0.1215 ^[6]	0.13769 ^[11]	0.13237 ^[9]	0.13296 ^[10]
		$\hat{\mu}$	0.00531 ^[4]	0.00571 ^[5]	0.00857 ^[10]	0.00407 ^[1]	0.00789 ^[9]	0.01161 ^[12]	0.00645 ^[8]	0.00504 ^[2]	0.0058 ^[6]	0.00507 ^[3]	0.00603 ^[7]	0.00943 ^[11]
	MRE	$\hat{\tau}$	0.2895 ^[1]	0.30982 ^[4]	0.31948 ^[7]	0.30921 ^[3]	0.3278 ^[8]	0.35714 ^[12]	0.31766 ^[6]	0.30562 ^[2]	0.31376 ^[5]	0.33328 ^[11]	0.32816 ^[9.5]	0.32816 ^[9.5]
$\sum Ranks$			11 ^[1]	26 ^[4]	47 ^[8]	16 ^[2.5]	50 ^[9]	70 ^[12]	40 ^[6]	16 ^[2.5]	34 ^[5]	41 ^[7]	53.5 ^[10]	63.5 ^[11]
150	BIAS	$\hat{\tau}$	0.21103 ^[1]	0.24864 ^[4]	0.26087 ^[9]	0.2487 ^[5]	0.2637 ^[10]	0.28281 ^[12]	0.2411 ^[3]	0.2367 ^[2]	0.26049 ^[7]	0.25802 ^[6]	0.26084 ^[8]	0.27256 ^[11]
		$\hat{\mu}$	0.04295 ^[1]	0.04996 ^[7]	0.05823 ^[10]	0.04437 ^[2]	0.05642 ^[9]	0.06939 ^[12]	0.05063 ^[8]	0.04865 ^[4]	0.04946 ^[5]	0.04734 ^[3]	0.04969 ^[6]	0.0657 ^[11]
	MSE	$\hat{\tau}$	0.07293 ^[1]	0.09949 ^[5]	0.10441 ^[6]	0.09892 ^[4]	0.11128 ^[10]	0.11848 ^[12]	0.09185 ^[3]	0.08759 ^[2]	0.10805 ^[8]	0.11082 ^[9]	0.10665 ^[7]	0.11233 ^[11]
		$\hat{\mu}$	0.00297 ^[2]	0.00389 ^[7]	0.00544 ^[10]	0.00293 ^[1]	0.00507 ^[9]	0.00757 ^[12]	0.00413 ^[8]	0.00366 ^[4]	0.00386 ^[5]	0.00345 ^[3]	0.00388 ^[6]	0.0068 ^[11]
	MRE	$\hat{\tau}$	0.23448 ^[1]	0.27627 ^[4]	0.28986 ^[10]	0.27633 ^[5]	0.293 ^[11]	0.31423 ^[12]	0.26789 ^[3]	0.263 ^[2]	0.28943 ^[7]	0.28669 ^[6]	0.28982 ^[8.5]	0.28982 ^[8.5]
$\sum Ranks$			7 ^[1]	33 ^[6]	54 ^[9]	19 ^[3]	57 ^[10]	70 ^[12]	32 ^[5]	18 ^[2]	37 ^[7]	30 ^[4]	47.5 ^[8]	63.5 ^[11]
200	BIAS	$\hat{\tau}$	0.19151 ^[1]	0.22403 ^[4]	0.23077 ^[6]	0.2283 ^[5]	0.24678 ^[9]	0.25087 ^[11]	0.21238 ^[3]	0.20718 ^[2]	0.23437 ^[7]	0.24416 ^[8]	0.25042 ^[10]	0.25244 ^[12]
		$\hat{\mu}$	0.03811 ^[1]	0.04421 ^[4]	0.05062 ^[9]	0.04031 ^[2]	0.05238 ^[10]	0.05967 ^[12]	0.0449 ^[3]	0.04144 ^[3]	0.04448 ^[6]	0.04446 ^[5]	0.04719 ^[8]	0.05679 ^[11]
	MSE	$\hat{\tau}$	0.05876 ^[1]	0.08232 ^[4]	0.08514 ^[5]	0.08721 ^[6]	0.09699 ^[9]	0.09677 ^[8]	0.07112 ^[2]	0.07126 ^[3]	0.08931 ^[7]	0.09715 ^[10]	0.1009 ^[12]	0.09795 ^[11]
		$\hat{\mu}$	0.00231 ^[1]	0.0031 ^[5]	0.00404 ^[9]	0.00241 ^[2]	0.00429 ^[10]	0.00564 ^[12]	0.00317 ^[7]	0.00272 ^[3]	0.00314 ^[6]	0.00295 ^[4]	0.00336 ^[8]	0.005 ^[11]
	MRE	$\hat{\tau}$	0.21279 ^[1]	0.24892 ^[4]	0.25641 ^[6]	0.25367 ^[5]	0.2742 ^[9]	0.27874 ^[12]	0.23598 ^[3]	0.2302 ^[2]	0.26041 ^[7]	0.27129 ^[8]	0.27825 ^[10.5]	0.27825 ^[10.5]
$\sum Ranks$			6 ^[1]	25 ^[4]	43 ^[8]	22 ^[3]	56 ^[9]	65 ^[11]	29 ^[5]	16 ^[2]	39 ^[6]	40 ^[7]	60.5 ^[10]	66.5 ^[12]
275	BIAS	$\hat{\tau}$	0.16499 ^[1]	0.20324 ^[5]	0.20593 ^[7]	0.19649 ^[4]	0.22145 ^[11]	0.243 ^[12]	0.19348 ^[3]	0.18376 ^[2]	0.21914 ^[10]	0.20409 ^[6]	0.21321 ^[8]	0.21484 ^[9]
		$\hat{\mu}$	0.03362 ^[1]	0.03963 ^[6]	0.04445 ^[9]	0.03458 ^[2]	0.04714 ^[11]	0.05269 ^[12]	0.03913 ^[5]	0.03595 ^[3]	0.04076 ^[8]	0.03626 ^[4]	0.03969 ^[7]	0.04686 ^[10]
	MSE	$\hat{\tau}$	0.04348 ^[1]	0.06907 ^[5]	0.06957 ^[6]	0.06748 ^[4]	0.07844 ^[10]	0.09379 ^[12]	0.06047 ^[3]	0.05516 ^[2]	0.08001 ^[11]	0.07281 ^[7]	0.07422 ^[8]	0.07444 ^[9]
		$\hat{\mu}$	0.0018 ^[1]	0.00239 ^[6]	0.00317 ^[9]	0.00186 ^[2]	0.0034 ^[11]	0.0042 ^[12]	0.0023 ^[5]	0.00199 ^[3]	0.00257 ^[8]	0.00206 ^[4]	0.00246 ^[7]	0.00339 ^[10]
	MRE	$\hat{\tau}$	0.18332 ^[1]	0.22583 ^[5]	0.22881 ^[7]	0.21832 ^[4]	0.24606 ^[11]	0.27 ^[12]	0.21498 ^[3]	0.20417 ^[2]	0.24349 ^[10]	0.22677 ^[6]	0.2369 ^[8.5]	0.2369 ^[8.5]
$\sum Ranks$			6 ^[1]	33 ^[6]	46 ^[7]	18 ^[3]	63 ^[11]	70 ^[12]	24 ^[4]	15 ^[2]	54 ^[9]	31 ^[5]	50.5 ^[8]	57.5 ^[10]
350	BIAS	$\hat{\tau}$	0.1482 ^[1]	0.17579 ^[5]	0.18975 ^[7]	0.16159 ^[2]	0.20541 ^[11]	0.22238 ^[12]	0.17454 ^[4]	0.16794 ^[3]	0.19403 ^[8]	0.18278 ^[6]	0.20444 ^[9]	0.20516 ^[10]
		$\hat{\mu}$	0.02903 ^[1]	0.03419 ^[5]	0.04054 ^[9]	0.02979 ^[2]	0.04194 ^[10]	0.04846 ^[12]	0.03506 ^[6]	0.03183 ^[3]	0.03637 ^[7]	0.03344 ^[4]	0.03712 ^[8]	0.04466 ^[11]
	MSE	$\hat{\tau}$	0.03619 ^[1]	0.05224 ^[5]	0.05755 ^[6]	0.04466 ^[2]	0.07113 ^[11]	0.08208 ^[12]	0.05058 ^[4]	0.04832 ^[3]	0.06322 ^[8]	0.05766 ^[7]	0.06901 ^[9]	0.06919 ^[10]
		$\hat{\mu}$	0.00135 ^[1]	0.00183 ^[5]	0.00255 ^[9]	0.0014 ^[2]	0.00278 ^[10]	0.00377 ^[12]	0.00197 ^[6]	0.0016 ^[3]	0.00202 ^[7]	0.00172 ^[4]	0.00212 ^[8]	0.00313 ^[11]
	MRE	$\hat{\tau}$	0.16466 ^[1]	0.19532 ^[5]	0.21084 ^[7]	0.17954 ^[2]	0.22824 ^[11]	0.24709 ^[12]	0.19394 ^[4]	0.18661 ^[3]	0.21559 ^[8]	0.20309 ^[6]	0.22716 ^[9.5]	0.22716 ^[9.5]
$\sum Ranks$			6 ^[1]	30 ^[4.5]	46 ^[8]	12 ^[2]	62 ^[10]	70 ^[12]	30 ^[4.5]	18 ^[3]	45 ^[7]	31 ^[6]	55.5 ^[9]	62.5 ^[11]
450	BIAS	$\hat{\tau}$	0.12887 ^[1]	0.14773 ^[4]	0.17128 ^[7]	0.14453 ^[2]	0.1747 ^[8]	0.19993 ^[12]	0.15438 ^[5]	0.14515 ^[3]	0.17913 ^[9]	0.16669 ^[6]	0.18805 ^[10]	0.18921 ^[11]
		$\hat{\mu}$	0.02577 ^[1]	0.02955 ^[4]	0.03541 ^[9]	0.02637 ^[2]	0.03594 ^[10]	0.04182 ^[12]	0.03108 ^[6]	0.02811 ^[3]	0.03304 ^[7]	0.03034 ^[5]	0.0339 ^[8]	0.04071 ^[11]
	MSE	$\hat{\tau}$	0.02661 ^[1]	0.03704 ^[4]	0.04985 ^[7]	0.03633 ^[3]	0.05092 ^[8]	0.06801 ^[12]	0.03899 ^[5]	0.03516 ^[2]	0.0551 ^[9]	0.04707 ^[6]	0.06027 ^[10]	0.06038 ^[11]
		$\hat{\mu}$	0.00104 ^[1]	0.00137 ^[4]	0.00195 ^[9.5]	0.00109 ^[2]	0.00195 ^[9.5]	0.00279 ^[12]	0.00148 ^[6]	0.00126 ^[3]	0.0017 ^[7]	0.00143 ^[5]	0.00176 ^[8]	0.00252 ^[11]
	MRE	$\hat{\tau}$	0.14319 ^[1]	0.16415 ^[4]	0.19031 ^[7]	0.16059 ^[2]	0.19411 ^[8]	0.22214 ^[12]	0.17153 ^[5]	0.16128 ^[3]	0.19904 ^[9]	0.18521 ^[6]	0.20895 ^[10.5]	0.20895 ^[10.5]
$\sum Ranks$			6 ^[1]	24 ^[4]	47.5 ^[7]	13 ^[2]	52.5 ^[9]	70 ^[12]	33 ^[5.5]	17 ^[3]	48 ^[8]	33 ^[5.5]	58.5 ^[10]	65.5 ^[11]
600	BIAS	$\hat{\tau}$	0.11416 ^[1]	0.1355 ^[4]	0.1517 ^[7]	0.12951 ^[3]	0.15333 ^[9]	0.17641 ^[12]	0.14219 ^[5]	0.12342 ^[2]	0.15224 ^[8]	0.14742 ^[6]	0.16492 ^[11]	0.15899 ^[10]
		$\hat{\mu}$	0.02255 ^[1]	0.02695 ^[4.5]	0.03131 ^[10]	0.02384 ^[2]	0.03122 ^[9]	0.03722 ^[12]	0.02784 ^[6]	0.02486 ^[3]	0.02875 ^[7]	0.02695 ^[4.5]	0.02968 ^[8]	0.03341 ^[11]
	MSE	$\hat{\tau}$	0.0202 ^[1]	0.0301 ^[4]	0.03863 ^[7]	0.02917 ^[3]	0.04025 ^[9]	0.05334 ^[12]	0.03335 ^[5]	0.0258 ^[2]	0.03876 ^[8]	0.03659 ^[6]	0.04842 ^[11]	0.04211 ^[10]
		$\hat{\mu}$	0.00079 ^[1]	0.00113 ^[5]	0.00152 ^[9]	0.00089 ^[2]	0.00153 ^[10]	0.00215 ^[12]	0.00119 ^[6]	0.00098 ^[3]	0.0013 ^[7]	0.00111 ^[4]	0.00139 ^[8]	0.00174 ^[11]
	MRE	$\hat{\tau}$	0.12685 ^[1]	0.15056 ^[4]	0.16855 ^[7]	0.14391 ^[3]	0.17037 ^[9]	0.19601 ^[12]	0.15799 ^[5]	0.13714 ^[2]	0.16916 ^[8]	0.16379 ^[6]	0.18325 ^[10.5]	0.18325 ^[10.5]
$\sum Ranks$			6 ^[1]	26 ^[4]	49 ^[8]	15 ^[2.5]	54 ^[9]	70 ^[12]	33 ^[6]	15 ^[2.5]	45 ^[7]	31 ^[5]	60.5 ^[10]	63.5 ^[11]

TABLE II. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 2.0$ and $\mu = 0.8$.

n	Est.	Est. Par.	EM ₁	EM ₂	EM ₃	EM ₄	EM ₅	EM ₆	EM ₇	EM ₈	EM ₉	EM ₁₀	EM ₁₁	EM ₁₂
25	BIAS	$\hat{\tau}$	0.75742 ^[1]	0.76594 ^[3]	0.77572 ^[8]	0.77366 ^[6]	0.78082 ^[9]	0.7811 ^[10]	0.77226 ^[5]	0.76446 ^[2]	0.77089 ^[4]	0.7756 ^[7]	0.78668 ^[11]	0.79528 ^[12]
		$\hat{\mu}$	0.30046 ^[9]	0.28935 ^[6]	0.30712 ^[10]	0.2665 ^[1]	0.29659 ^[8]	0.31651 ^[11]	0.28801 ^[5]	0.29092 ^[7]	0.27142 ^[2]	0.27208 ^[3]	0.28436 ^[4]	0.32386 ^[12]
	MSE	$\hat{\tau}$	0.76338 ^[1]	0.79154 ^[3]	0.80193 ^[4]	0.81849 ^[8]	0.81681 ^[7]	0.81362 ^[6]	0.80778 ^[5]	0.78763 ^[2]	0.82394 ^[9]	0.83027 ^[11]	0.83693 ^[12]	0.83018 ^[10]
		$\hat{\mu}$	0.13715 ^[9]	0.12735 ^[6]	0.14125 ^[10]	0.10762 ^[1]	0.13191 ^[8]	0.14777 ^[11]	0.12562 ^[5]	0.12832 ^[7]	0.11328 ^[3]	0.11301 ^[2]	0.12374 ^[4]	0.15183 ^[12]
	MRE	$\hat{\tau}$	0.37871 ^[1]	0.38297 ^[3]	0.38786 ^[8]	0.38683 ^[6]	0.39041 ^[9]	0.39055 ^[10]	0.38613 ^[5]	0.38223 ^[2]	0.38545 ^[4]	0.3878 ^[7]	0.39334 ^[11.5]	0.39334 ^[11.5]
		$\hat{\mu}$	0.37558 ^[8]	0.36169 ^[5]	0.3839 ^[9]	0.33313 ^[1]	0.37074 ^[7]	0.39564 ^[10]	0.36001 ^[4]	0.36365 ^[6]	0.33927 ^[2]	0.3401 ^[3]	1.68814 ^[12]	0.39764 ^[11]
Σ Ranks			29 ^[5.5]	26 ^[3.5]	49 ^[9]	23 ^[1]	48 ^[8]	58 ^[11]	29 ^[5.5]	26 ^[3.5]	24 ^[2]	33 ^[7]	54.5 ^[10]	68.5 ^[12]
90	BIAS	$\hat{\tau}$	0.6538 ^[2]	0.67816 ^[4]	0.70075 ^[10]	0.68122 ^[6]	0.69833 ^[9]	0.71951 ^[11]	0.68095 ^[5]	0.64834 ^[1]	0.67387 ^[3]	0.69466 ^[8]	0.69244 ^[7]	0.72176 ^[12]
		$\hat{\mu}$	0.22409 ^[5]	0.23038 ^[7]	0.26032 ^[10]	0.19727 ^[1]	0.25162 ^[9]	0.27985 ^[11]	0.23772 ^[8]	0.21393 ^[4]	0.21209 ^[3]	0.21129 ^[2]	0.22572 ^[6]	0.28086 ^[12]
	MSE	$\hat{\tau}$	0.61227 ^[1]	0.65253 ^[3]	0.68539 ^[6]	0.6962 ^[9]	0.68569 ^[7]	0.71149 ^[11]	0.66158 ^[4]	0.61777 ^[2]	0.67191 ^[5]	0.7062 ^[10]	0.68757 ^[8]	0.71162 ^[12]
		$\hat{\mu}$	0.0801 ^[5]	0.0826 ^[7]	0.10511 ^[10]	0.0597 ^[1]	0.09809 ^[9]	0.1192 ^[12]	0.08907 ^[8]	0.07308 ^[4]	0.07229 ^[3]	0.06908 ^[2]	0.08092 ^[6]	0.11809 ^[11]
	MRE	$\hat{\tau}$	0.3269 ^[2]	0.33908 ^[4]	0.35037 ^[11]	0.34061 ^[6]	0.34917 ^[10]	0.35976 ^[12]	0.34047 ^[5]	0.32417 ^[1]	0.33693 ^[3]	0.34733 ^[9]	0.34622 ^[7.5]	0.34622 ^[7.5]
		$\hat{\mu}$	0.28011 ^[5]	0.28798 ^[6]	0.3254 ^[9]	0.24659 ^[1]	0.31453 ^[8]	0.34981 ^[10]	0.29715 ^[7]	0.26742 ^[4]	0.26512 ^[3]	0.26411 ^[2]	1.82427 ^[12]	0.36088 ^[11]
Σ Ranks			20 ^[2.5]	31 ^[5]	56 ^[10]	24 ^[4]	52 ^[9]	67 ^[12]	37 ^[7]	16 ^[1]	20 ^[2.5]	33 ^[6]	46.5 ^[8]	65.5 ^[11]
150	BIAS	$\hat{\tau}$	0.56047 ^[1]	0.6211 ^[5]	0.64318 ^[8]	0.617 ^[3]	0.6482 ^[10]	0.68614 ^[12]	0.62086 ^[4]	0.6081 ^[2]	0.63464 ^[7]	0.647 ^[9]	0.62252 ^[6]	0.67695 ^[11]
		$\hat{\mu}$	0.18034 ^[3]	0.19748 ^[7]	0.23143 ^[10]	0.16713 ^[1]	0.22073 ^[9]	0.25694 ^[12]	0.19936 ^[8]	0.18805 ^[5]	0.1884 ^[6]	0.17955 ^[2]	0.18453 ^[4]	0.24635 ^[11]
	MSE	$\hat{\tau}$	0.474 ^[1]	0.57551 ^[3]	0.5953 ^[6]	0.59971 ^[7]	0.61956 ^[9]	0.66077 ^[12]	0.57634 ^[4]	0.55348 ^[2]	0.61686 ^[8]	0.63726 ^[10]	0.5871 ^[5]	0.65795 ^[11]
		$\hat{\mu}$	0.05251 ^[3]	0.06128 ^[7]	0.08483 ^[10]	0.04239 ^[1]	0.0773 ^[9]	0.10253 ^[12]	0.06348 ^[8]	0.05554 ^[5]	0.05601 ^[6]	0.04964 ^[2]	0.05388 ^[4]	0.09294 ^[11]
	MRE	$\hat{\tau}$	0.28023 ^[1]	0.31055 ^[5]	0.32159 ^[9]	0.3085 ^[3]	0.3241 ^[11]	0.34307 ^[12]	0.31043 ^[4]	0.30405 ^[2]	0.31732 ^[8]	0.3235 ^[10]	0.31126 ^[6.5]	0.31126 ^[6.5]
		$\hat{\mu}$	0.22543 ^[3]	0.24685 ^[6]	0.28929 ^[9]	0.20891 ^[1]	0.27592 ^[8]	0.32118 ^[10]	0.2492 ^[7]	0.23506 ^[4]	0.2355 ^[5]	0.22443 ^[2]	1.90985 ^[12]	0.33847 ^[11]
Σ Ranks			12 ^[1]	33 ^[4]	52 ^[9]	16 ^[2]	56 ^[10]	70 ^[12]	35 ^[5.5]	20 ^[3]	40 ^[8]	35 ^[5.5]	37.5 ^[7]	61.5 ^[11]
200	BIAS	$\hat{\tau}$	0.50977 ^[1]	0.57875 ^[3]	0.61561 ^[9]	0.58429 ^[4]	0.62704 ^[10]	0.6582 ^[12]	0.59242 ^[6]	0.55568 ^[2]	0.58469 ^[5]	0.60122 ^[8]	0.59375 ^[7]	0.64285 ^[11]
		$\hat{\mu}$	0.15375 ^[2]	0.17541 ^[7]	0.21239 ^[10]	0.15244 ^[1]	0.20305 ^[9]	0.24307 ^[12]	0.18294 ^[8]	0.16726 ^[4]	0.16828 ^[5]	0.16306 ^[3]	0.1685 ^[6]	0.22999 ^[11]
	MSE	$\hat{\tau}$	0.40702 ^[1]	0.51773 ^[3]	0.56089 ^[8]	0.5549 ^[7]	0.58977 ^[10]	0.61525 ^[12]	0.53826 ^[5]	0.47763 ^[2]	0.53813 ^[4]	0.57221 ^[9]	0.55273 ^[6]	0.60083 ^[11]
		$\hat{\mu}$	0.03772 ^[2]	0.04854 ^[7]	0.07192 ^[10]	0.03495 ^[1]	0.0641 ^[9]	0.09185 ^[12]	0.05226 ^[8]	0.04382 ^[4]	0.04536 ^[6]	0.04032 ^[3]	0.04487 ^[5]	0.08209 ^[11]
	MRE	$\hat{\tau}$	0.25489 ^[1]	0.28938 ^[3]	0.30781 ^[10]	0.29215 ^[4]	0.31352 ^[11]	0.3291 ^[12]	0.29621 ^[6]	0.27784 ^[2]	0.29235 ^[5]	0.30061 ^[9]	0.29688 ^[7.5]	0.29688 ^[7.5]
		$\hat{\mu}$	0.19218 ^[2]	0.21927 ^[6]	0.26548 ^[9]	0.19055 ^[1]	0.25382 ^[8]	0.30383 ^[10]	0.22867 ^[7]	0.20908 ^[4]	0.21034 ^[5]	0.20382 ^[3]	1.94347 ^[12]	0.32142 ^[11]
Σ Ranks			9 ^[1]	29 ^[4]	56 ^[9]	18 ^[2.5]	57 ^[10]	70 ^[12]	40 ^[7]	18 ^[2.5]	30 ^[5]	35 ^[6]	43.5 ^[8]	62.5 ^[11]
275	BIAS	$\hat{\tau}$	0.4571 ^[1]	0.52248 ^[4]	0.57531 ^[9]	0.52032 ^[3]	0.58112 ^[10]	0.62814 ^[12]	0.53557 ^[5]	0.49382 ^[2]	0.56514 ^[8]	0.55681 ^[7]	0.55257 ^[6]	0.60887 ^[11]
		$\hat{\mu}$	0.13567 ^[2]	0.15554 ^[7]	0.18822 ^[10]	0.13285 ^[1]	0.1868 ^[9]	0.21876 ^[12]	0.16005 ^[8]	0.14492 ^[3]	0.15512 ^[6]	0.14723 ^[4]	0.14979 ^[5]	0.209 ^[11]
	MSE	$\hat{\tau}$	0.33348 ^[1]	0.44229 ^[3]	0.50601 ^[8]	0.46013 ^[5]	0.51489 ^[9]	0.58503 ^[12]	0.45635 ^[4]	0.39429 ^[2]	0.52034 ^[10]	0.50401 ^[7]	0.49128 ^[6]	0.55531 ^[11]
		$\hat{\mu}$	0.02918 ^[2]	0.03807 ^[7]	0.056 ^[10]	0.02679 ^[1]	0.0551 ^[9]	0.07497 ^[12]	0.0399 ^[8]	0.033 ^[4]	0.03769 ^[6]	0.03275 ^[3]	0.03489 ^[5]	0.06726 ^[11]
	MRE	$\hat{\tau}$	0.22855 ^[1]	0.26124 ^[4]	0.28766 ^[10]	0.26016 ^[3]	0.29056 ^[11]	0.31407 ^[12]	0.26779 ^[5]	0.24691 ^[2]	0.28257 ^[9]	0.2784 ^[8]	0.27629 ^[6.5]	0.27629 ^[6.5]
		$\hat{\mu}$	0.16958 ^[2]	0.19442 ^[6]	0.23527 ^[9]	0.16606 ^[1]	0.2335 ^[8]	0.27345 ^[10]	0.20006 ^[7]	0.18116 ^[3]	0.1939 ^[5]	0.18403 ^[4]	1.98399 ^[12]	0.30444 ^[11]
Σ Ranks			9 ^[1]	31 ^[4]	56 ^[9.5]	14 ^[2]	56 ^[9.5]	70 ^[12]	37 ^[6]	16 ^[3]	44 ^[8]	33 ^[5]	40.5 ^[7]	61.5 ^[11]
350	BIAS	$\hat{\tau}$	0.40923 ^[1]	0.48124 ^[3]	0.54405 ^[9]	0.48224 ^[4]	0.54531 ^[10]	0.5883 ^[12]	0.49931 ^[5]	0.45544 ^[2]	0.51605 ^[7]	0.52197 ^[8]	0.51145 ^[6]	0.56923 ^[11]
		$\hat{\mu}$	0.11904 ^[1]	0.14141 ^[7]	0.17169 ^[10]	0.12357 ^[2]	0.16629 ^[9]	0.19801 ^[12]	0.14639 ^[8]	0.12995 ^[3]	0.14098 ^[6]	0.13532 ^[4]	0.13795 ^[5]	0.18521 ^[11]
	MSE	$\hat{\tau}$	0.27532 ^[1]	0.382 ^[3]	0.4637 ^[9]	0.39691 ^[4]	0.47773 ^[10]	0.52862 ^[12]	0.40688 ^[5]	0.34193 ^[2]	0.44138 ^[7]	0.45026 ^[8]	0.43121 ^[6]	0.4962 ^[11]
		$\hat{\mu}$	0.02284 ^[1]	0.03108 ^[6]	0.04607 ^[10]	0.02297 ^[2]	0.04281 ^[9]	0.06172 ^[12]	0.03395 ^[8]	0.02637 ^[3]	0.03117 ^[7]	0.02729 ^[4]	0.02943 ^[5]	0.0526 ^[11]
	MRE	$\hat{\tau}$	0.20461 ^[1]	0.24062 ^[3]	0.27203 ^[10]	0.24112 ^[4]	0.27265 ^[11]	0.29415 ^[12]	0.24966 ^[5]	0.22772 ^[2]	0.25803 ^[8]	0.26099 ^[9]	0.25572 ^[6.5]	0.25572 ^[6.5]
		$\hat{\mu}$	0.1488 ^[1]	0.17676 ^[6]	0.21461 ^[9]	0.15447 ^[2]	0.20786 ^[8]	0.24751 ^[10]	0.18299 ^[7]	0.16244 ^[3]	0.17623 ^[5]	0.16915 ^[4]	2.02323 ^[12]	0.28461 ^[11]
Σ Ranks			6 ^[1]	28 ^[4]	57 ^[9.5]	18 ^[3]	57 ^[9.5]	70 ^[12]	38 ^[6]	15 ^[2]	40 ^[7]	37 ^[5]	40.5 ^[8]	61.5 ^[11]
450	BIAS	$\hat{\tau}$	0.37296 ^[1]	0.44276 ^[4]	0.49916 ^[9]	0.42497 ^[3]	0.50761 ^[10]	0.56058 ^[12]	0.44683 ^[5]	0.41894 ^[2]	0.48639 ^[7]	0.47468 ^[6]	0.48791 ^[8]	0.54087 ^[11]
		$\hat{\mu}$	0.10606 ^[1]	0.12986 ^[7]	0.15526 ^[10]	0.10955 ^[2]	0.15409 ^[9]	0.18151 ^[12]	0.12962 ^[6]	0.11763 ^[3]	0.13096 ^[8]	0.1234 ^[4]	0.12811 ^[5]	0.173 ^[11]
	MSE	$\hat{\tau}$	0.23081 ^[1]	0.32302 ^[4]	0.39978 ^[8]	0.31521 ^[3]	0.4151 ^[10]	0.49628 ^[12]	0.3271 ^[5]	0.29676 ^[2]	0.40027 ^[9]	0.38407 ^[6]	0.39372 ^[7]	0.46378 ^[11]
		$\hat{\mu}$	0.01756 ^[1]	0.026 ^[7]	0.03804 ^[10]	0.01838 ^[2]	0.03708 ^[9]	0.05187 ^[12]	0.02591 ^[6]	0.02152 ^[3]	0.02629 ^[8]	0.02297 ^[4]	0.02483 ^[5]	0.04564 ^[11]
	MRE	$\hat{\tau}$	0.18648 ^[1]	0.22138 ^[4]	0.249598									

TABLE III. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 1.5$ and $\mu = 2.5$.

n		Est.	Est. Par.	EM ₁	EM ₂	EM ₃	EM ₄	EM ₅	EM ₆	EM ₇	EM ₈	EM ₉	EM ₁₀	EM ₁₁	EM ₁₂
25	BIAS	$\hat{\tau}$		0.57829 ^[3]	0.57684 ^[2]	0.5852 ^[6]	0.58044 ^[4]	0.58745 ^[7]	0.60508 ^[12]	0.60244 ^[11]	0.59596 ^[9]	0.57015 ^[1]	0.59745 ^[10]	0.58763 ^[8]	0.58297 ^[5]
		$\hat{\mu}$		0.88238 ^[6]	0.88429 ^[9]	0.90761 ^[10]	0.78948 ^[1]	0.88401 ^[8]	0.94578 ^[11]	0.8824 ^[7]	0.8617 ^[4]	0.8114 ^[2]	0.82163 ^[3]	0.86414 ^[5]	0.97608 ^[12]
	MSE	$\hat{\tau}$		0.45094 ^[2]	0.45235 ^[3]	0.46746 ^[7]	0.47333 ^[8]	0.46708 ^[6]	0.47774 ^[10]	0.48705 ^[12]	0.4759 ^[9]	0.4552 ^[4]	0.48145 ^[11]	0.46539 ^[5]	0.44699 ^[1]
		$\hat{\mu}$		1.22547 ^[9]	1.20378 ^[8]	1.25989 ^[10]	0.9867 ^[1]	1.18133 ^[6]	1.33149 ^[11]	1.20183 ^[7]	1.13378 ^[4]	1.02165 ^[2]	1.03084 ^[3]	1.16211 ^[5]	1.42032 ^[12]
	MRE	$\hat{\tau}$		0.38552 ^[3]	0.38456 ^[2]	0.39013 ^[5]	0.38696 ^[4]	0.39163 ^[6]	0.40338 ^[12]	0.40163 ^[11]	0.3973 ^[9]	0.3801 ^[1]	0.3983 ^[10]	0.39175 ^[7.5]	0.39175 ^[7.5]
		$\hat{\mu}$		0.35295 ^[5]	0.35371 ^[8]	0.36304 ^[9]	0.31579 ^[1]	0.35361 ^[7]	0.37831 ^[10]	0.35296 ^[6]	0.34468 ^[4]	0.32456 ^[2]	0.32865 ^[3]	1.26142 ^[12]	0.38865 ^[11]
90	BIAS	$\hat{\tau}$		28 ^[3]	32 ^[4]	47 ^[9]	19 ^[2]	40 ^[6.5]	66 ^[12]	54 ^[11]	39 ^[5]	12 ^[1]	40 ^[6.5]	42.5 ^[8]	48.5 ^[10]
		$\hat{\mu}$		0.46099 ^[2]	0.48791 ^[3]	0.53115 ^[11]	0.49071 ^[4]	0.52879 ^[10]	0.5208 ^[9]	0.5082 ^[6]	0.44932 ^[1]	0.51584 ^[7]	0.51672 ^[8]	0.50309 ^[5]	0.54003 ^[12]
	MSE	$\hat{\tau}$		0.57225 ^[2]	0.6277 ^[6]	0.76667 ^[11]	0.54315 ^[1]	0.73233 ^[9]	0.7654 ^[10]	0.67055 ^[8]	0.58383 ^[3]	0.6186 ^[5]	0.59884 ^[4]	0.64107 ^[7]	0.81462 ^[12]
		$\hat{\mu}$		0.31015 ^[2]	0.35596 ^[3]	0.39618 ^[10]	0.36902 ^[5]	0.3938 ^[9]	0.38401 ^[7]	0.37116 ^[6]	0.30274 ^[1]	0.39279 ^[8]	0.40211 ^[11]	0.36667 ^[4]	0.40664 ^[12]
	MRE	$\hat{\tau}$		0.52819 ^[2]	0.62324 ^[6]	0.94599 ^[11]	0.45172 ^[1]	0.81669 ^[9]	0.92306 ^[10]	0.72054 ^[8]	0.56409 ^[4]	0.59688 ^[5]	0.56163 ^[3]	0.65916 ^[7]	1.01422 ^[12]
		$\hat{\mu}$		0.30733 ^[2]	0.32527 ^[3]	0.3541 ^[12]	0.32714 ^[4]	0.35253 ^[11]	0.3472 ^[10]	0.3388 ^[7]	0.29954 ^[1]	0.34389 ^[8]	0.34448 ^[9]	0.3354 ^[5.5]	0.3354 ^[5.5]
150	BIAS	$\hat{\tau}$		0.2289 ^[2]	0.25108 ^[6]	0.30667 ^[10]	0.21726 ^[1]	0.29293 ^[8]	0.30616 ^[9]	0.26822 ^[7]	0.23353 ^[3]	0.24744 ^[5]	0.23954 ^[4]	1.3899 ^[12]	0.36002 ^[11]
		$\hat{\mu}$		12 ^[1]	27 ^[4]	65 ^[12]	16 ^[3]	56 ^[10]	55 ^[9]	42 ^[8]	13 ^[2]	38 ^[5]	39 ^[6]	40.5 ^[7]	64.5 ^[11]
	MSE	$\hat{\tau}$		0.39035 ^[1]	0.44757 ^[3]	0.46173 ^[8]	0.4536 ^[6]	0.4672 ^[9]	0.48674 ^[11]	0.45345 ^[5]	0.4166 ^[2]	0.45097 ^[4]	0.45695 ^[7]	0.46875 ^[10]	0.48833 ^[12]
		$\hat{\mu}$		0.46725 ^[2]	0.53727 ^[6]	0.63791 ^[10]	0.46466 ^[1]	0.6104 ^[9]	0.6933 ^[12]	0.55284 ^[8]	0.49777 ^[4]	0.51223 ^[5]	0.4935 ^[3]	0.54027 ^[7]	0.68973 ^[11]
	MRE	$\hat{\tau}$		0.24385 ^[1]	0.30825 ^[4]	0.31425 ^[5]	0.33338 ^[9]	0.32552 ^[7]	0.34348 ^[11]	0.30713 ^[3]	0.26987 ^[2]	0.31984 ^[6]	0.33154 ^[8]	0.33491 ^[10]	0.34381 ^[12]
		$\hat{\mu}$		0.36635 ^[3]	0.45352 ^[6]	0.65417 ^[10]	0.32341 ^[1]	0.59591 ^[9]	0.75993 ^[12]	0.4853 ^[8]	0.39503 ^[4]	0.42049 ^[5]	0.36356 ^[2]	0.4609 ^[7]	0.73638 ^[11]
200	BIAS	$\hat{\tau}$		0.26023 ^[1]	0.29838 ^[3]	0.30782 ^[8]	0.3024 ^[6]	0.31147 ^[9]	0.3245 ^[12]	0.3023 ^[5]	0.27773 ^[2]	0.30065 ^[4]	0.30463 ^[7]	0.3125 ^[10.5]	0.3125 ^[10.5]
		$\hat{\mu}$		0.1869 ^[2]	0.21491 ^[6]	0.25516 ^[9]	0.18586 ^[1]	0.24416 ^[8]	0.27732 ^[10]	0.22114 ^[7]	0.19911 ^[4]	0.20489 ^[5]	0.1974 ^[3]	1.44353 ^[12]	0.32555 ^[11]
	MSE	$\hat{\tau}$		10 ^[1]	28 ^[4]	50 ^[8]	24 ^[3]	51 ^[9]	68 ^[12]	36 ^[7]	18 ^[2]	29 ^[5]	30 ^[6]	56.5 ^[10]	67.5 ^[11]
		$\hat{\mu}$		0.36439 ^[1]	0.42291 ^[6]	0.44035 ^[9]	0.41596 ^[4]	0.44124 ^[10]	0.45634 ^[11]	0.41298 ^[3]	0.36843 ^[2]	0.42461 ^[7]	0.42098 ^[5]	0.43249 ^[8]	0.45661 ^[12]
	MRE	$\hat{\tau}$		0.41297 ^[1]	0.49908 ^[7]	0.57872 ^[10]	0.41527 ^[2]	0.53627 ^[9]	0.63311 ^[12]	0.49921 ^[8]	0.42866 ^[3]	0.46262 ^[5]	0.43841 ^[4]	0.47608 ^[6]	0.61672 ^[11]
		$\hat{\mu}$		0.21301 ^[1]	0.28513 ^[5]	0.29318 ^[8]	0.27913 ^[4]	0.3065 ^[10]	0.30859 ^[11]	0.26693 ^[3]	0.22062 ^[2]	0.29368 ^[9]	0.28673 ^[6]	0.29254 ^[7]	0.31664 ^[12]
275	BIAS	$\hat{\tau}$		0.27419 ^[2]	0.39133 ^[8]	0.53396 ^[10]	0.2589 ^[1]	0.45464 ^[9]	0.64025 ^[12]	0.38661 ^[7]	0.29805 ^[4]	0.33889 ^[5]	0.29211 ^[3]	0.35285 ^[6]	0.58861 ^[11]
		$\hat{\mu}$		0.24293 ^[1]	0.28194 ^[6]	0.29356 ^[10]	0.27731 ^[4]	0.29416 ^[11]	0.30423 ^[12]	0.27532 ^[3]	0.24562 ^[2]	0.28308 ^[7]	0.28066 ^[5]	0.28832 ^[8.5]	0.28832 ^[8.5]
	MSE	$\hat{\tau}$		0.16519 ^[1]	0.19963 ^[6]	0.23149 ^[9]	0.16611 ^[2]	0.21451 ^[8]	0.25324 ^[10]	0.19968 ^[7]	0.17146 ^[3]	0.18505 ^[5]	0.17536 ^[4]	1.505 ^[12]	0.30441 ^[11]
		$\hat{\mu}$		7 ^[1]	38 ^[6.5]	56 ^[9]	17 ^[3]	57 ^[10]	68 ^[12]	31 ^[5]	16 ^[2]	38 ^[6.5]	27 ^[4]	47.5 ^[8]	65.5 ^[11]
	MRE	$\hat{\tau}$		0.33319 ^[1]	0.37008 ^[4]	0.40546 ^[9]	0.35985 ^[3]	0.41711 ^[10]	0.43962 ^[12]	0.37482 ^[5]	0.34819 ^[2]	0.39943 ^[8]	0.38689 ^[6]	0.3992 ^[7]	0.41768 ^[11]
		$\hat{\mu}$		0.37727 ^[2]	0.4283 ^[6]	0.50174 ^[9]	0.36704 ^[1]	0.50502 ^[10]	0.58503 ^[12]	0.43684 ^[8]	0.38833 ^[4]	0.43296 ^[7]	0.38364 ^[3]	0.41181 ^[5]	0.54046 ^[11]
350	BIAS	$\hat{\tau}$		0.17781 ^[1]	0.22331 ^[5]	0.25573 ^[8]	0.21958 ^[3]	0.2772 ^[11]	0.29567 ^[12]	0.22118 ^[4]	0.19522 ^[2]	0.25558 ^[7]	0.25353 ^[6]	0.25845 ^[9]	0.2674 ^[10]
		$\hat{\mu}$		0.21936 ^[2]	0.27844 ^[6]	0.39312 ^[9]	0.20133 ^[1]	0.40704 ^[10]	0.54133 ^[12]	0.3034 ^[8]	0.23428 ^[4]	0.28837 ^[7]	0.22905 ^[3]	0.25701 ^[5]	0.45305 ^[11]
	MSE	$\hat{\tau}$		0.22213 ^[1]	0.24672 ^[4]	0.27031 ^[10]	0.2399 ^[3]	0.27807 ^[11]	0.29308 ^[12]	0.24988 ^[5]	0.23212 ^[2]	0.26629 ^[9]	0.25793 ^[6]	0.26613 ^[7.5]	0.26613 ^[7.5]
		$\hat{\mu}$		0.15091 ^[2]	0.17132 ^[5]	0.20069 ^[8]	0.14681 ^[1]	0.20201 ^[9]	0.23401 ^[10]	0.17474 ^[7]	0.15533 ^[4]	0.17318 ^[6]	0.15345 ^[3]	1.51551 ^[12]	0.27845 ^[11]
	MRE	$\hat{\tau}$		9 ^[1]	30 ^[5]	53 ^[9]	12 ^[2]	61 ^[10]	70 ^[12]	37 ^[6]	18 ^[3]	44 ^[7]	27 ^[4]	45.5 ^[8]	61.5 ^[11]
		$\hat{\mu}$		0.29368 ^[1]	0.34103 ^[4]	0.38375 ^[10]	0.33216 ^[2]	0.38273 ^[9]	0.42535 ^[12]	0.34541 ^[5]	0.3359 ^[3]	0.37274 ^[8]	0.36475 ^[6]	0.36672 ^[7]	0.388 ^[11]
450	BIAS	$\hat{\tau}$		0.32415 ^[1]	0.38823 ^[7]	0.47174 ^[10]	0.33528 ^[2]	0.44496 ^[9]	0.54104 ^[12]	0.3834 ^[6]	0.35923 ^[3]	0.39161 ^[8]	0.3693 ^[5]	0.36787 ^[4]	0.4925 ^[11]
		$\hat{\mu}$		0.14732 ^[1]	0.19128 ^[4]	0.23256 ^[8]	0.18913 ^[3]	0.23714 ^[10]	0.28615 ^[12]	0.20155 ^[5]	0.1891 ^[2]	0.23381 ^[9]	0.22159 ^[6]	0.22868 ^[7]	0.23871 ^[11]
	MSE	$\hat{\tau}$		0.16565 ^[1]	0.23257 ^[7]	0.34073 ^[10]	0.16831 ^[2]	0.3048 ^[9]	0.45665 ^[12]	0.23009 ^[6]	0.20493 ^[3]	0.24015 ^[8]	0.2092 ^[5]	0.20728 ^[4]	0.37306 ^[11]
		$\hat{\mu}$		0.19578 ^[1]	0.22735 ^[4]	0.25584 ^[11]	0.22144 ^[2]	0.25516 ^[10]	0.28356 ^[12]	0.23028 ^[5]	0.22393 ^[3]	0.24849 ^[7]	0.24316 ^[6]	0.24448 ^[7.5]	0.24448 ^[7.5]
	MRE	$\hat{\tau}$		0.12966 ^[1]	0.15529 ^[6]	0.18869 ^[9]	0.13411 ^[2]	0.17798 ^[8]	0.21641 ^[10]	0.15336 ^[5]	0.14369 ^[3]	0.15664 ^[7]	0.14772 ^[4]	1.53374 ^[12]	0.25867 ^[11]
		$\hat{\mu}$		6 ^[1]	32 ^[5]	58 ^[10]	13 ^[2]	55 ^[9]	70 ^[12]	32 ^[5]	17 ^[3]	49 ^[8]	32 ^[5]	41.5 ^[7]	62.5 ^[11]
600	BIAS	$\hat{\tau}$		0.24843 ^[1]	0.29856 ^[5]	0.33882 ^[7]	0.28813 ^[3]	0.34837 ^[10]	0.389 ^[12]	0.29593 ^[4]	0.28741 ^[2]	0.34125 ^[8]	0.3259 ^[6]	0.34226 ^[9]	0.36725 ^[11]
		$\hat{\mu}$		0.2763 ^[1]	0.34124 ^[7]	0.41153 ^[10]	0.29631 ^[2]	0.4042 ^[9]	0.49202 ^[12]	0.33323 ^[5]	0.31392 ^[3]	0.35901 ^[8]	0.32799 ^[4]	0.33984 ^[6]	0.45338 ^[11]
	MSE	$\hat{\tau}$		0.10878 ^[1]	0.14685 ^[5]	0.18123 ^[6]	0.14011 ^[2]	0.20561 ^[10]	0.23934 ^[12]	0.14569 ^[4]	0.14323 ^[3]	0.19739 ^[8]	0.18359 ^[7]	0.20445 ^[9]	

TABLE IV. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 0.75$ and $\mu = 1.5$.

n	Est.	Est. Par.	EM ₁	EM ₂	EM ₃	EM ₄	EM ₅	EM ₆	EM ₇	EM ₈	EM ₉	EM ₁₀	EM ₁₁	EM ₁₂
25	BIAS	$\hat{\tau}$	0.2945 ^[4]	0.29244 ^[2]	0.30394 ^[8]	0.29089 ^[11]	0.29929 ^[7]	0.31252 ^[12]	0.29555 ^[5]	0.29345 ^[3]	0.30754 ^[11]	0.30575 ^[10]	0.2965 ^[6]	0.30563 ^[9]
		$\hat{\mu}$	0.46901 ^[9]	0.44312 ^[7]	0.50681 ^[10]	0.37913 ^[11]	0.45012 ^[8]	0.51407 ^[11]	0.43586 ^[5]	0.4356 ^[4]	0.42332 ^[2]	0.42663 ^[3]	0.44157 ^[6]	0.52579 ^[12]
	MSE	$\hat{\tau}$	0.11644 ^[1]	0.11817 ^[2]	0.12443 ^[9]	0.12047 ^[4]	0.12228 ^[6]	0.12976 ^[11]	0.12055 ^[5]	0.11837 ^[3]	0.12982 ^[12]	0.12476 ^[10]	0.12339 ^[7]	0.12406 ^[8]
		$\hat{\mu}$	0.35205 ^[9]	0.32072 ^[7]	0.40308 ^[10]	0.23388 ^[11]	0.32724 ^[8]	0.41345 ^[11]	0.31186 ^[4]	0.31415 ^[5]	0.29186 ^[2]	0.29375 ^[3]	0.31976 ^[6]	0.4309 ^[12]
	MRE	$\hat{\tau}$	0.39266 ^[4]	0.38992 ^[2]	0.40525 ^[9]	0.38785 ^[11]	0.39906 ^[8]	0.4167 ^[12]	0.39406 ^[5]	0.39127 ^[3]	0.41005 ^[11]	0.40767 ^[10]	0.39533 ^[6.5]	0.39533 ^[6.5]
		$\hat{\mu}$	0.31267 ^[8]	0.29541 ^[6]	0.33787 ^[9]	0.25275 ^[11]	0.30008 ^[7]	0.34272 ^[10]	0.29057 ^[5]	0.2904 ^[4]	0.28221 ^[2]	0.28442 ^[3]	0.63929 ^[12]	0.4075 ^[11]
$\sum Ranks$			35 ^[5]	26 ^[3]	55 ^[10]	9 ^[1]	44 ^[9]	67 ^[12]	29 ^[4]	22 ^[2]	40 ^[7]	39 ^[6]	43.5 ^[8]	58.5 ^[11]
90	BIAS	$\hat{\tau}$	0.21114 ^[11]	0.23366 ^[3]	0.24031 ^[7]	0.23382 ^[4]	0.24971 ^[10]	0.25485 ^[12]	0.2371 ^[5]	0.21627 ^[2]	0.24862 ^[9]	0.24061 ^[8]	0.23971 ^[6]	0.25317 ^[11]
		$\hat{\mu}$	0.26495 ^[3]	0.29027 ^[6]	0.34 ^[10]	0.24528 ^[11]	0.31841 ^[9]	0.37669 ^[12]	0.29691 ^[8]	0.26838 ^[4]	0.29554 ^[7]	0.26423 ^[2]	0.28782 ^[5]	0.35323 ^[11]
	MSE	$\hat{\tau}$	0.06696 ^[1]	0.08545 ^[3]	0.08475 ^[5]	0.08584 ^[6]	0.09338 ^[12]	0.09316 ^[10]	0.08512 ^[4]	0.07015 ^[2]	0.08993 ^[9]	0.08725 ^[7]	0.08787 ^[8]	0.09333 ^[11]
		$\hat{\mu}$	0.11539 ^[4]	0.13843 ^[6]	0.19487 ^[10]	0.08913 ^[11]	0.15403 ^[9]	0.23114 ^[12]	0.1433 ^[8]	0.11416 ^[3]	0.13913 ^[7]	0.10603 ^[2]	0.13632 ^[5]	0.19937 ^[11]
	MRE	$\hat{\tau}$	0.28152 ^[1]	0.31154 ^[3]	0.32041 ^[8]	0.31176 ^[4]	0.33295 ^[11]	0.3398 ^[12]	0.31614 ^[5]	0.28836 ^[2]	0.3315 ^[10]	0.32081 ^[9]	0.31961 ^[6.5]	0.31961 ^[6.5]
		$\hat{\mu}$	0.17663 ^[3]	0.19351 ^[5]	0.22667 ^[9]	0.16352 ^[11]	0.21227 ^[8]	0.25113 ^[10]	0.19794 ^[7]	0.17892 ^[4]	0.19703 ^[6]	0.17616 ^[12]	0.74183 ^[12]	0.33756 ^[11]
$\sum Ranks$			13 ^[1]	28 ^[4]	47 ^[8]	17 ^[2.5]	59 ^[10]	68 ^[12]	37 ^[6]	17 ^[2.5]	48 ^[9]	30 ^[5]	42.5 ^[7]	61.5 ^[11]
150	BIAS	$\hat{\tau}$	0.18152 ^[11]	0.18334 ^[2]	0.21329 ^[6]	0.19694 ^[4]	0.20533 ^[5]	0.22952 ^[12]	0.21384 ^[7]	0.18863 ^[3]	0.22874 ^[11]	0.22082 ^[8]	0.22231 ^[9]	0.2224 ^[10]
		$\hat{\mu}$	0.21138 ^[2]	0.21715 ^[4]	0.27356 ^[10]	0.20021 ^[11]	0.26002 ^[9]	0.31315 ^[12]	0.24641 ^[7]	0.21625 ^[3]	0.24246 ^[6]	0.22769 ^[5]	0.24793 ^[8]	0.29698 ^[11]
	MSE	$\hat{\tau}$	0.05219 ^[1]	0.05435 ^[2]	0.07053 ^[6]	0.0652 ^[4]	0.0665 ^[5]	0.07833 ^[11]	0.07188 ^[7]	0.05621 ^[3]	0.08205 ^[12]	0.07775 ^[10]	0.07692 ^[9]	0.07252 ^[8]
		$\hat{\mu}$	0.06943 ^[2]	0.0753 ^[4]	0.11934 ^[10]	0.06146 ^[11]	0.10893 ^[9]	0.1552 ^[12]	0.09426 ^[7]	0.07386 ^[3]	0.08858 ^[6]	0.07769 ^[5]	0.0979 ^[8]	0.13585 ^[11]
	MRE	$\hat{\tau}$	0.24203 ^[1]	0.24445 ^[2]	0.28439 ^[6]	0.26259 ^[4]	0.27378 ^[5]	0.30602 ^[12]	0.28511 ^[7]	0.25151 ^[3]	0.30499 ^[11]	0.29443 ^[8]	0.29641 ^[9.5]	0.29641 ^[9.5]
		$\hat{\mu}$	0.14092 ^[2]	0.14477 ^[4]	0.18237 ^[9]	0.13347 ^[11]	0.17335 ^[8]	0.20877 ^[10]	0.16427 ^[7]	0.14416 ^[3]	0.16164 ^[6]	0.15179 ^[5]	0.73603 ^[12]	0.29654 ^[11]
$\sum Ranks$			9 ^[1]	18 ^[3.5]	47 ^[8]	15 ^[2]	41 ^[5.5]	69 ^[12]	42 ^[7]	18 ^[3.5]	52 ^[9]	41 ^[5.5]	55.5 ^[10]	60.5 ^[11]
200	BIAS	$\hat{\tau}$	0.15667 ^[11]	0.17459 ^[3]	0.19837 ^[8]	0.18141 ^[4]	0.2009 ^[9]	0.21253 ^[12]	0.18278 ^[5]	0.16839 ^[2]	0.20968 ^[11]	0.19456 ^[6]	0.1954 ^[7]	0.20641 ^[10]
		$\hat{\mu}$	0.18219 ^[2]	0.20588 ^[5]	0.2461 ^[10]	0.18096 ^[11]	0.24151 ^[9]	0.27568 ^[12]	0.21302 ^[7]	0.18693 ^[3]	0.22356 ^[8]	0.1994 ^[4]	0.20599 ^[6]	0.25596 ^[11]
	MSE	$\hat{\tau}$	0.039 ^[1]	0.04915 ^[3]	0.0627 ^[7]	0.05694 ^[5]	0.06519 ^[9]	0.07069 ^[12]	0.05291 ^[4]	0.04575 ^[2]	0.06898 ^[11]	0.06129 ^[6]	0.06415 ^[8]	0.06699 ^[10]
		$\hat{\mu}$	0.05212 ^[2]	0.06559 ^[5]	0.09656 ^[10]	0.04961 ^[11]	0.09033 ^[9]	0.12005 ^[12]	0.07126 ^[7]	0.05464 ^[3]	0.07854 ^[8]	0.05894 ^[4]	0.0666 ^[6]	0.09853 ^[11]
	MRE	$\hat{\tau}$	0.20889 ^[1]	0.23279 ^[3]	0.2645 ^[9]	0.24188 ^[4]	0.26787 ^[10]	0.28337 ^[12]	0.24371 ^[5]	0.22452 ^[2]	0.27957 ^[11]	0.25941 ^[6]	0.26053 ^[7.5]	0.26053 ^[7.5]
		$\hat{\mu}$	0.12146 ^[2]	0.13725 ^[5]	0.16407 ^[9]	0.12064 ^[11]	0.16101 ^[8]	0.18379 ^[10]	0.14201 ^[6]	0.12462 ^[3]	0.14904 ^[7]	0.13293 ^[4]	0.77329 ^[12]	0.27521 ^[11]
$\sum Ranks$			9 ^[1]	24 ^[4]	53 ^[8]	16 ^[3]	54 ^[9]	70 ^[12]	34 ^[6]	15 ^[2]	56 ^[10]	30 ^[5]	46.5 ^[7]	60.5 ^[11]
275	BIAS	$\hat{\tau}$	0.13216 ^[11]	0.14988 ^[4]	0.17118 ^[7]	0.14926 ^[3]	0.17711 ^[9]	0.19025 ^[12]	0.15548 ^[5]	0.14502 ^[2]	0.1796 ^[10]	0.16627 ^[6]	0.17707 ^[8]	0.18662 ^[11]
		$\hat{\mu}$	0.14713 ^[1]	0.17088 ^[4]	0.21018 ^[10]	0.15602 ^[2]	0.20417 ^[9]	0.24166 ^[12]	0.17657 ^[6]	0.16601 ^[3]	0.18937 ^[8]	0.17296 ^[5]	0.18541 ^[7]	0.2343 ^[11]
	MSE	$\hat{\tau}$	0.02938 ^[1]	0.03755 ^[3]	0.04621 ^[6]	0.03934 ^[4]	0.05273 ^[9]	0.05925 ^[12]	0.04051 ^[5]	0.035 ^[2]	0.05418 ^[10]	0.04711 ^[7]	0.05124 ^[8]	0.05562 ^[11]
		$\hat{\mu}$	0.03447 ^[1]	0.04642 ^[4]	0.06727 ^[10]	0.03773 ^[2]	0.06448 ^[9]	0.09131 ^[12]	0.04936 ^[6]	0.04391 ^[3]	0.05505 ^[8]	0.04668 ^[5]	0.05324 ^[7]	0.08294 ^[11]
	MRE	$\hat{\tau}$	0.17621 ^[1]	0.19984 ^[4]	0.22823 ^[7]	0.19901 ^[3]	0.23614 ^[10]	0.25367 ^[12]	0.20731 ^[5]	0.19335 ^[2]	0.23947 ^[11]	0.22169 ^[6]	0.23609 ^[8.5]	0.23609 ^[8.5]
		$\hat{\mu}$	0.09809 ^[1]	0.11392 ^[4]	0.14012 ^[9]	0.10401 ^[2]	0.13611 ^[8]	0.16111 ^[10]	0.11771 ^[6]	0.11067 ^[3]	0.12625 ^[7]	0.11531 ^[5]	0.77002 ^[12]	0.24882 ^[11]
$\sum Ranks$			6 ^[1]	23 ^[4]	49 ^[7]	16 ^[3]	54 ^[9.5]	70 ^[12]	33 ^[5]	15 ^[2]	54 ^[9.5]	34 ^[6]	50.5 ^[8]	63.5 ^[11]
350	BIAS	$\hat{\tau}$	0.11669 ^[11]	0.13595 ^[4]	0.14615 ^[6]	0.13175 ^[2]	0.16724 ^[11]	0.17736 ^[12]	0.14176 ^[5]	0.1333 ^[3]	0.15771 ^[9]	0.15264 ^[7]	0.15376 ^[8]	0.16088 ^[10]
		$\hat{\mu}$	0.13237 ^[1]	0.15414 ^[4]	0.1798 ^[9]	0.13967 ^[2]	0.1953 ^[10]	0.22232 ^[12]	0.16087 ^[6]	0.1521 ^[3]	0.16704 ^[8]	0.15778 ^[5]	0.16482 ^[7]	0.19824 ^[11]
	MSE	$\hat{\tau}$	0.02195 ^[1]	0.03252 ^[4]	0.03404 ^[6]	0.02874 ^[2]	0.04639 ^[11]	0.05195 ^[12]	0.03317 ^[5]	0.02926 ^[3]	0.04241 ^[10]	0.04021 ^[7]	0.04105 ^[8]	0.04235 ^[9]
		$\hat{\mu}$	0.02788 ^[1]	0.0385 ^[4]	0.05132 ^[9]	0.0299 ^[2]	0.05738 ^[10]	0.0778 ^[12]	0.03982 ^[6]	0.03647 ^[3]	0.04422 ^[8]	0.03869 ^[5]	0.04324 ^[7]	0.06004 ^[11]
	MRE	$\hat{\tau}$	0.15559 ^[1]	0.18127 ^[4]	0.19487 ^[6]	0.17566 ^[2]	0.22298 ^[11]	0.23648 ^[12]	0.18901 ^[5]	0.17774 ^[3]	0.21028 ^[10]	0.20352 ^[7]	0.20501 ^[8.5]	0.20501 ^[8.5]
		$\hat{\mu}$	0.08824 ^[1]	0.10276 ^[4]	0.11987 ^[8]	0.09311 ^[2]	0.1302 ^[9]	0.14821 ^[10]	0.10725 ^[6]	0.1014 ^[3]	0.11136 ^[7]	0.10519 ^[5]	0.76666 ^[12]	0.21451 ^[11]
$\sum Ranks$			6 ^[1]	24 ^[4]	44 ^[7]	12 ^[2]	62 ^[11]	70 ^[12]	33 ^[5]	18 ^[3]	52 ^[9]	36 ^[6]	50.5 ^[8]	60.5 ^[10]
450	BIAS	$\hat{\tau}$	0.1075 ^[11]	0.11633 ^[4]	0.13839 ^[7]	0.11202 ^[3]	0.14059 ^[9]	0.16006 ^[12]	0.12503 ^[5]	0.11077 ^[2]	0.13985 ^[8]	0.13513 ^[6]	0.1411 ^[10]	0.14927 ^[11]
		$\hat{\mu}$	0.12033 ^[1]	0.13517 ^[4]	0.16705 ^[9]	0.12035 ^[2]	0.16753 ^[10]	0.19687 ^[12]	0.14464 ^[6]	0.12766 ^[3]	0.14942 ^[8]	0.13937 ^[5]	0.14842 ^[7]	0.18485 ^[11]
	MSE	$\hat{\tau}$	0.01846 ^[1]	0.02341 ^[4]	0.0312 ^[6]	0.02112 ^[3]	0.033 ^[8.5]	0.04106 ^[12]	0.02501 ^[5]	0.02 ^[2]	0.033 ^[8.5]	0.03181 ^[7]	0.03352 ^[10]	0.03458 ^[11]
		$\hat{\mu}$	0.02319 ^[2]	0.02931 ^[4]	0.0433 ^[10]	0.02273 ^[11]	0.04315 ^[9]	0.06071 ^[12]	0.0318 ^[6]	0.02548 ^[3]	0.03458 ^[7]	0.02988 ^[5]	0.03495 ^[8]	0.05076 ^[11]
	MRE	$\hat{\tau}$	0											

TABLE V. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 2.5$ and $\mu = 0.25$.

n	Est.	Est. Par.	EM ₁	EM ₂	EM ₃	EM ₄	EM ₅	EM ₆	EM ₇	EM ₈	EM ₉	EM ₁₀	EM ₁₁	EM ₁₂
25	BIAS	$\hat{\tau}$	0.94904 ^[1]	0.95793 ^[5]	0.96406 ^[8]	0.96314 ^[7]	0.96293 ^[6]	0.95532 ^[3]	0.96886 ^[11]	0.95508 ^[2]	0.95705 ^[4]	0.96568 ^[9]	0.96606 ^[10]	0.99548 ^[12]
		$\hat{\mu}$	0.09649 ^[9]	0.0949 ^[8]	0.09968 ^[11]	0.08639 ^[1]	0.09314 ^[6]	0.09775 ^[10]	0.09344 ^[7]	0.09095 ^[4]	0.08805 ^[2]	0.08926 ^[3]	0.09235 ^[5]	0.10166 ^[12]
	MSE	$\hat{\tau}$	1.2013 ^[1]	1.231 ^[3]	1.2379 ^[5]	1.26425 ^[8]	1.24862 ^[6]	1.22133 ^[2]	1.26911 ^[9]	1.23751 ^[4]	1.27508 ^[10]	1.281 ^[11]	1.25331 ^[7]	1.31707 ^[12]
		$\hat{\mu}$	0.01399 ^[9]	0.01343 ^[8]	0.01465 ^[11]	0.01125 ^[1]	0.01286 ^[6]	0.01417 ^[10]	0.01297 ^[7]	0.01251 ^[4]	0.0117 ^[2]	0.01195 ^[3]	0.01283 ^[5]	0.01485 ^[12]
	MRE	$\hat{\tau}$	0.37962 ^[1]	0.38317 ^[5]	0.38562 ^[8]	0.38525 ^[7]	0.38517 ^[6]	0.38213 ^[3]	0.38754 ^[12]	0.38203 ^[2]	0.38282 ^[4]	0.38627 ^[9]	0.38643 ^[10.5]	0.38643 ^[10.5]
		$\hat{\mu}$	0.38596 ^[8]	0.3796 ^[7]	0.39873 ^[11]	0.34555 ^[1]	0.37255 ^[5]	0.39099 ^[9]	0.37377 ^[6]	0.36381 ^[4]	0.3522 ^[2]	0.35705 ^[3]	2.14353 ^[12]	0.39819 ^[10]
$\sum Ranks$			29 ^[4]	36 ^[6]	54 ^[11]	25 ^[3]	35 ^[5]	37 ^[7]	52 ^[10]	20 ^[1]	24 ^[2]	38 ^[8]	49.5 ^[9]	68.5 ^[12]
90	BIAS	$\hat{\tau}$	0.82617 ^[1]	0.86588 ^[6]	0.88548 ^[9]	0.84684 ^[3]	0.8887 ^[10]	0.90574 ^[11]	0.86691 ^[7]	0.83671 ^[2]	0.87234 ^[8]	0.86474 ^[4]	0.86577 ^[5]	0.92548 ^[12]
		$\hat{\mu}$	0.07614 ^[6]	0.07936 ^[8]	0.08674 ^[10]	0.06672 ^[1]	0.08408 ^[9]	0.09212 ^[12]	0.07832 ^[7]	0.07426 ^[4]	0.07216 ^[3]	0.06992 ^[2]	0.07432 ^[5]	0.09199 ^[11]
	MSE	$\hat{\tau}$	0.96384 ^[1]	1.04938 ^[3]	1.0814 ^[7]	1.07111 ^[6]	1.09271 ^[8]	1.11092 ^[11]	1.06612 ^[5]	0.99825 ^[2]	1.10021 ^[10]	1.09387 ^[9]	1.06321 ^[4]	1.16682 ^[12]
		$\hat{\mu}$	0.00906 ^[6]	0.00975 ^[8]	0.01152 ^[10]	0.0069 ^[1]	0.01076 ^[9]	0.01278 ^[12]	0.00962 ^[7]	0.00862 ^[5]	0.00827 ^[3]	0.00766 ^[2]	0.00857 ^[4]	0.01249 ^[11]
	MRE	$\hat{\tau}$	0.33047 ^[1]	0.34635 ^[7]	0.35419 ^[10]	0.33874 ^[3]	0.35548 ^[11]	0.3623 ^[12]	0.34676 ^[8]	0.33469 ^[2]	0.34894 ^[9]	0.3459 ^[4]	0.34631 ^[5.5]	0.34631 ^[5.5]
		$\hat{\mu}$	0.30457 ^[5]	0.31746 ^[7]	0.34697 ^[9]	0.26687 ^[1]	0.3363 ^[8]	0.36849 ^[10]	0.31326 ^[6]	0.29702 ^[4]	0.28862 ^[3]	0.2797 ^[2]	2.29677 ^[12]	0.37019 ^[11]
$\sum Ranks$			20 ^[3]	39 ^[7]	55 ^[9.5]	15 ^[1]	55 ^[9.5]	68 ^[12]	40 ^[8]	19 ^[2]	36 ^[6]	23 ^[4]	35.5 ^[5]	62.5 ^[11]
150	BIAS	$\hat{\tau}$	0.73392 ^[1]	0.79553 ^[4]	0.85498 ^[11]	0.79059 ^[3]	0.82466 ^[9]	0.85404 ^[10]	0.80865 ^[6]	0.77248 ^[2]	0.8038 ^[5]	0.8164 ^[8]	0.81277 ^[7]	0.87033 ^[12]
		$\hat{\mu}$	0.06242 ^[3]	0.0683 ^[7]	0.08115 ^[10]	0.05708 ^[1]	0.07523 ^[9]	0.08455 ^[12]	0.07011 ^[8]	0.06514 ^[6]	0.06245 ^[4]	0.06135 ^[2]	0.06478 ^[5]	0.08403 ^[11]
	MSE	$\hat{\tau}$	0.80084 ^[1]	0.93138 ^[3]	1.0229 ^[11]	0.97714 ^[7]	0.9764 ^[6]	1.01722 ^[10]	0.95464 ^[4]	0.88006 ^[2]	0.98231 ^[8]	1.01175 ^[9]	0.97259 ^[5]	1.06669 ^[12]
		$\hat{\mu}$	0.00625 ^[4]	0.00737 ^[7]	0.01005 ^[10]	0.00498 ^[1]	0.00886 ^[9]	0.01098 ^[12]	0.00777 ^[8]	0.00669 ^[6]	0.00615 ^[3]	0.00583 ^[2]	0.00655 ^[5]	0.01075 ^[11]
	MRE	$\hat{\tau}$	0.29357 ^[1]	0.31821 ^[4]	0.34199 ^[12]	0.31624 ^[3]	0.32987 ^[10]	0.34162 ^[11]	0.32346 ^[6]	0.30899 ^[2]	0.32152 ^[5]	0.32656 ^[9]	0.32511 ^[7.5]	0.32511 ^[7.5]
		$\hat{\mu}$	0.24967 ^[3]	0.27319 ^[6]	0.32461 ^[9]	0.2283 ^[1]	0.30091 ^[8]	0.33819 ^[10]	0.28045 ^[7]	0.26056 ^[5]	0.24981 ^[4]	0.24541 ^[2]	2.3516 ^[12]	0.34813 ^[11]
$\sum Ranks$			13 ^[1]	31 ^[5]	63 ^[10]	16 ^[2]	51 ^[9]	65 ^[12]	39 ^[7]	23 ^[3]	29 ^[4]	32 ^[6]	41.5 ^[8]	64.5 ^[11]
200	BIAS	$\hat{\tau}$	0.67648 ^[1]	0.75419 ^[3]	0.79076 ^[9]	0.75535 ^[4]	0.80715 ^[10]	0.84112 ^[11]	0.76407 ^[5]	0.73687 ^[2]	0.77652 ^[7]	0.775 ^[6]	0.77796 ^[8]	0.85248 ^[12]
		$\hat{\mu}$	0.05517 ^[2]	0.06242 ^[7]	0.07341 ^[10]	0.05231 ^[1]	0.07112 ^[9]	0.08145 ^[12]	0.06312 ^[8]	0.0583 ^[4]	0.059 ^[6]	0.05571 ^[3]	0.05837 ^[5]	0.07895 ^[11]
	MSE	$\hat{\tau}$	0.70936 ^[1]	0.8566 ^[3]	0.90423 ^[5]	0.91415 ^[6]	0.94758 ^[10]	0.99435 ^[11]	0.88638 ^[4]	0.83933 ^[2]	0.92609 ^[7]	0.93603 ^[9]	0.9294 ^[8]	1.02893 ^[12]
		$\hat{\mu}$	0.00489 ^[3]	0.00614 ^[7]	0.00842 ^[10]	0.00414 ^[1]	0.00789 ^[9]	0.01024 ^[12]	0.00637 ^[8]	0.00534 ^[4.5]	0.00551 ^[6]	0.00479 ^[2]	0.00534 ^[4.5]	0.00933 ^[11]
	MRE	$\hat{\tau}$	0.27059 ^[1]	0.30168 ^[3]	0.3163 ^[10]	0.30214 ^[4]	0.32286 ^[11]	0.33645 ^[12]	0.30563 ^[5]	0.29475 ^[2]	0.31061 ^[7]	0.31 ^[6]	0.31118 ^[8.5]	0.31118 ^[8.5]
		$\hat{\mu}$	0.22067 ^[2]	0.24969 ^[6]	0.29366 ^[9]	0.20924 ^[1]	0.28446 ^[8]	0.32578 ^[10]	0.25249 ^[7]	0.23318 ^[4]	0.23601 ^[5]	0.22286 ^[3]	2.42697 ^[12]	0.34099 ^[11]
$\sum Ranks$			10 ^[1]	29 ^[4.5]	53 ^[9]	17 ^[2]	57 ^[10]	68 ^[12]	37 ^[6]	18.5 ^[3]	38 ^[7]	29 ^[4.5]	46 ^[8]	65.5 ^[11]
275	BIAS	$\hat{\tau}$	0.60887 ^[1]	0.71047 ^[5]	0.73768 ^[9]	0.6872 ^[3]	0.75695 ^[10]	0.80779 ^[11]	0.71466 ^[6]	0.67699 ^[2]	0.72766 ^[8]	0.70953 ^[4]	0.72001 ^[7]	0.81514 ^[12]
		$\hat{\mu}$	0.0482 ^[2]	0.05625 ^[7]	0.06545 ^[10]	0.04634 ^[1]	0.06359 ^[9]	0.07557 ^[12]	0.05703 ^[8]	0.05162 ^[4]	0.05382 ^[6]	0.0497 ^[3]	0.05194 ^[5]	0.07206 ^[11]
	MSE	$\hat{\tau}$	0.58592 ^[1]	0.78185 ^[4]	0.80751 ^[6]	0.78139 ^[3]	0.86561 ^[10]	0.94032 ^[11]	0.7903 ^[5]	0.72897 ^[2]	0.83212 ^[9]	0.81629 ^[7]	0.81688 ^[8]	0.97619 ^[12]
		$\hat{\mu}$	0.00372 ^[2]	0.0049 ^[7]	0.00674 ^[10]	0.00324 ^[1]	0.00628 ^[9]	0.00887 ^[12]	0.00512 ^[8]	0.00415 ^[4]	0.00458 ^[6]	0.0038 ^[3]	0.00416 ^[5]	0.00789 ^[11]
	MRE	$\hat{\tau}$	0.24355 ^[1]	0.28419 ^[5]	0.29507 ^[10]	0.27488 ^[3]	0.30278 ^[11]	0.32312 ^[12]	0.28586 ^[6]	0.2708 ^[2]	0.29106 ^[9]	0.28381 ^[4]	0.288 ^[7.5]	0.288 ^[7.5]
		$\hat{\mu}$	0.19279 ^[2]	0.225 ^[6]	0.26179 ^[9]	0.18538 ^[1]	0.25436 ^[8]	0.30229 ^[10]	0.22811 ^[7]	0.20646 ^[4]	0.21529 ^[5]	0.1988 ^[3]	2.49663 ^[12]	0.32606 ^[11]
$\sum Ranks$			9 ^[1]	34 ^[5]	54 ^[9]	12 ^[2]	57 ^[10]	68 ^[12]	40 ^[6]	18 ^[3]	43 ^[7]	24 ^[4]	44.5 ^[8]	64.5 ^[11]
350	BIAS	$\hat{\tau}$	0.56105 ^[1]	0.65403 ^[4]	0.7088 ^[9]	0.6212 ^[3]	0.72122 ^[10]	0.78125 ^[12]	0.6583 ^[5]	0.61238 ^[2]	0.69514 ^[8]	0.69176 ^[7]	0.68721 ^[6]	0.76031 ^[11]
		$\hat{\mu}$	0.04346 ^[2]	0.05002 ^[7]	0.06079 ^[10]	0.04222 ^[1]	0.05947 ^[9]	0.06977 ^[12]	0.05124 ^[8]	0.04642 ^[3]	0.04954 ^[6]	0.04715 ^[4]	0.0482 ^[5]	0.06583 ^[11]
	MSE	$\hat{\tau}$	0.51034 ^[1]	0.69059 ^[4]	0.76914 ^[7]	0.6557 ^[3]	0.80953 ^[10]	0.90136 ^[12]	0.69477 ^[5]	0.61166 ^[2]	0.79033 ^[9]	0.78987 ^[8]	0.76135 ^[6]	0.86673 ^[11]
		$\hat{\mu}$	0.00301 ^[2]	0.00384 ^[7]	0.00585 ^[10]	0.00266 ^[1]	0.0056 ^[9]	0.0076 ^[12]	0.00415 ^[8]	0.00337 ^[3.5]	0.00382 ^[6]	0.00337 ^[3.5]	0.00353 ^[5]	0.00663 ^[11]
	MRE	$\hat{\tau}$	0.22442 ^[1]	0.26161 ^[4]	0.28352 ^[10]	0.24848 ^[3]	0.28849 ^[11]	0.3125 ^[12]	0.26332 ^[5]	0.24495 ^[2]	0.27806 ^[9]	0.2767 ^[8]	0.27489 ^[6.5]	0.27489 ^[6.5]
		$\hat{\mu}$	0.17385 ^[2]	0.20007 ^[6]	0.24316 ^[9]	0.16889 ^[1]	0.23789 ^[8]	0.27909 ^[10]	0.20495 ^[7]	0.1857 ^[3]	0.19817 ^[5]	0.18861 ^[4]	2.51047 ^[12]	0.30412 ^[11]
$\sum Ranks$			9 ^[1]	32 ^[4]	55 ^[9]	12 ^[2]	57 ^[10]	70 ^[12]	38 ^[6]	15.5 ^[3]	43 ^[8]	34.5 ^[5]	40.5 ^[7]	61.5 ^[11]
450	BIAS	$\hat{\tau}$	0.50917 ^[1]	0.60178 ^[4]	0.68103 ^[10]	0.57542 ^[3]	0.6731 ^[9]	0.74305 ^[12]	0.61054 ^[5]	0.55884 ^[2]	0.64563 ^[8]	0.6337 ^[7]	0.63185 ^[6]	0.72157 ^[11]
		$\hat{\mu}$	0.03852 ^[2]	0.04517 ^[7]	0.05585 ^[10]	0.03836 ^[1]	0.05339 ^[9]	0.06446 ^[12]	0.04699 ^[8]	0.04219 ^[3]	0.04506 ^[6]	0.04277 ^[4]	0.04357 ^[5]	0.06132 ^[11]
	MSE	$\hat{\tau}$	0.42971 ^[1]	0.60509 ^[5]	0.7333 ^[10]	0.56894 ^[3]	0.72641 ^[9]	0.83492 ^[12]	0.6018 ^[4]	0.52085 ^[2]	0.70307 ^[8]	0.67391 ^[7]	0.66557 ^[6]	0.79812 ^[11]
		$\hat{\mu}$	0.00233 ^[2]	0.00319 ^[7]	0.00491 ^[10]	0.00222 ^[1]	0.00444 ^[9]	0.00651 ^[12]	0.00342 ^[8]	0.00278 ^[4]	0.00315 ^[6]	0.00276 ^[3]	0.0029 ^[5]	0.00573 ^[11]
	MRE	$\hat{\tau}$	0.20											

TABLE VI. Numerical values of simulation measures (BIAS, MSE, and MRE) for $\tau = 0.3$ and $\mu = 3.5$.

n	Est.	Est. Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\tau}$	0.1203 ^[5]	0.1232 ^[7]	0.12563 ^[9]	0.11889 ^[2]	0.11974 ^[4]	0.12876 ^[12]	0.11602 ^[11]	0.12031 ^[6]	0.11903 ^[3]	0.12323 ^[8]	0.12744 ^[10]	0.12768 ^[11]
		$\hat{\mu}$	0.85234 ^[7]	0.85245 ^[8]	0.9502 ^[10]	0.72492 ^[11]	0.82218 ^[5]	1.05033 ^[12]	0.76838 ^[2]	0.82296 ^[6]	0.79462 ^[4]	0.77576 ^[3]	0.87228 ^[9]	1.01617 ^[11]
	MSE	$\hat{\tau}$	0.0198 ^[3]	0.0209 ^[7]	0.02095 ^[8]	0.01995 ^[6]	0.01986 ^[5]	0.02212 ^[12]	0.01904 ^[11]	0.01982 ^[4]	0.01945 ^[2]	0.02121 ^[9]	0.02198 ^[10]	0.02202 ^[11]
		$\hat{\mu}$	1.25958 ^[8]	1.24082 ^[7]	1.53154 ^[10]	0.87626 ^[11]	1.14831 ^[5]	1.76609 ^[12]	1.01233 ^[2]	1.17671 ^[6]	1.07319 ^[4]	1.01973 ^[3]	1.31098 ^[9]	1.70714 ^[11]
	MRE	$\hat{\tau}$	0.40099 ^[5]	0.41066 ^[7]	0.41877 ^[9]	0.39629 ^[2]	0.39915 ^[4]	0.4292 ^[12]	0.38674 ^[11]	0.40105 ^[6]	0.39677 ^[3]	0.41076 ^[8]	0.4248 ^[10.5]	0.4248 ^[10.5]
		$\hat{\mu}$	0.24353 ^[7]	0.24356 ^[8]	0.27149 ^[10]	0.20712 ^[11]	0.23491 ^[5]	0.30009 ^[11]	0.21954 ^[2]	0.23513 ^[6]	0.22704 ^[4]	0.22164 ^[3]	0.25893 ^[9]	0.4256 ^[12]
$\sum Ranks$			35 ^[7]	44 ^[8]	56 ^[9]	13 ^[2]	28 ^[4]	71 ^[12]	9 ^[1]	34 ^[5.5]	20 ^[3]	34 ^[5.5]	57.5 ^[10]	66.5 ^[11]
90	BIAS	$\hat{\tau}$	0.07856 ^[11]	0.08493 ^[3]	0.08774 ^[5]	0.08807 ^[6]	0.09178 ^[7]	0.09864 ^[12]	0.08539 ^[4]	0.08357 ^[2]	0.09531 ^[10]	0.09469 ^[9]	0.09572 ^[11]	0.09243 ^[8]
		$\hat{\mu}$	0.42028 ^[11]	0.45831 ^[5]	0.52579 ^[10]	0.42063 ^[2]	0.5117 ^[9]	0.62162 ^[12]	0.44841 ^[4]	0.43534 ^[3]	0.48937 ^[7]	0.46626 ^[6]	0.50147 ^[8]	0.5528 ^[11]
	MSE	$\hat{\tau}$	0.0095 ^[11]	0.01145 ^[4]	0.01147 ^[5]	0.01223 ^[6]	0.01265 ^[7]	0.01415 ^[12]	0.01138 ^[3]	0.01105 ^[2]	0.01398 ^[11]	0.01374 ^[10]	0.01367 ^[9]	0.01299 ^[8]
		$\hat{\mu}$	0.28331 ^[2]	0.33113 ^[6]	0.46407 ^[10]	0.2604 ^[11]	0.41107 ^[9]	0.6244 ^[12]	0.31965 ^[4]	0.29763 ^[3]	0.38393 ^[7]	0.32937 ^[5]	0.40573 ^[8]	0.49072 ^[11]
	MRE	$\hat{\tau}$	0.26186 ^[11]	0.2831 ^[3]	0.29248 ^[5]	0.29358 ^[6]	0.30594 ^[7]	0.32879 ^[12]	0.28464 ^[4]	0.27856 ^[2]	0.31771 ^[9]	0.31565 ^[8]	0.31905 ^[10.5]	0.31905 ^[10.5]
		$\hat{\mu}$	0.12008 ^[11]	0.13095 ^[5]	0.15023 ^[9]	0.12018 ^[2]	0.1462 ^[8]	0.17761 ^[10]	0.12812 ^[4]	0.12438 ^[3]	0.13982 ^[7]	0.13322 ^[6]	0.29789 ^[11]	0.30811 ^[12]
$\sum Ranks$			7 ^[1]	26 ^[5]	44 ^[6.5]	23 ^[3.5]	47 ^[8]	70 ^[12]	23 ^[3.5]	15 ^[2]	51 ^[9]	44 ^[6.5]	57.5 ^[10]	60.5 ^[11]
150	BIAS	$\hat{\tau}$	0.06271 ^[11]	0.07177 ^[4]	0.07641 ^[7]	0.07176 ^[3]	0.07407 ^[5]	0.08336 ^[11]	0.07595 ^[6]	0.07079 ^[2]	0.0829 ^[10]	0.07971 ^[9]	0.08672 ^[12]	0.07709 ^[8]
		$\hat{\mu}$	0.32188 ^[11]	0.37142 ^[4]	0.42906 ^[10]	0.34173 ^[2]	0.41617 ^[8]	0.49163 ^[12]	0.40113 ^[7]	0.36539 ^[3]	0.39944 ^[6]	0.37582 ^[5]	0.4195 ^[9]	0.45072 ^[11]
	MSE	$\hat{\tau}$	0.00626 ^[11]	0.00832 ^[3]	0.00927 ^[7]	0.00851 ^[4]	0.0088 ^[5]	0.01069 ^[10]	0.00898 ^[6]	0.00789 ^[2]	0.01072 ^[11]	0.01031 ^[9]	0.01175 ^[12]	0.00961 ^[8]
		$\hat{\mu}$	0.16313 ^[11]	0.21892 ^[5]	0.28689 ^[10]	0.17503 ^[2]	0.27482 ^[9]	0.37889 ^[12]	0.24599 ^[6]	0.20733 ^[3]	0.24895 ^[7]	0.21666 ^[4]	0.27059 ^[8]	0.32804 ^[11]
	MRE	$\hat{\tau}$	0.20903 ^[11]	0.23923 ^[4]	0.25472 ^[7]	0.2392 ^[3]	0.24689 ^[5]	0.27788 ^[10]	0.25317 ^[6]	0.23597 ^[2]	0.27632 ^[9]	0.26569 ^[8]	0.28908 ^[11.5]	0.28908 ^[11.5]
		$\hat{\mu}$	0.09197 ^[11]	0.10612 ^[4]	0.12259 ^[9]	0.09764 ^[2]	0.11891 ^[8]	0.14046 ^[10]	0.11461 ^[7]	0.1044 ^[3]	0.11413 ^[6]	0.10738 ^[5]	0.30315 ^[12]	0.25696 ^[11]
$\sum Ranks$			6 ^[1]	24 ^[4]	50 ^[9]	16 ^[3]	40 ^[6.5]	65 ^[12]	38 ^[5]	15 ^[2]	49 ^[8]	40 ^[6.5]	64.5 ^[11]	60.5 ^[10]
200	BIAS	$\hat{\tau}$	0.05775 ^[11]	0.06228 ^[2]	0.07028 ^[8]	0.06271 ^[3]	0.06784 ^[7]	0.07207 ^[10]	0.06437 ^[4]	0.06448 ^[5]	0.07428 ^[12]	0.07037 ^[9]	0.07427 ^[11]	0.06681 ^[6]
		$\hat{\mu}$	0.29631 ^[2]	0.3233 ^[3]	0.38439 ^[11]	0.29238 ^[11]	0.35927 ^[9]	0.41755 ^[12]	0.34227 ^[6]	0.32941 ^[4]	0.35452 ^[8]	0.33675 ^[5]	0.34663 ^[7]	0.3839 ^[10]
	MSE	$\hat{\tau}$	0.00536 ^[11]	0.00643 ^[2]	0.00777 ^[8]	0.00658 ^[4]	0.00755 ^[7]	0.00862 ^[10]	0.00644 ^[3]	0.00659 ^[5]	0.00909 ^[11]	0.00826 ^[9]	0.00914 ^[12]	0.0073 ^[6]
		$\hat{\mu}$	0.14108 ^[2]	0.16734 ^[3]	0.22927 ^[10]	0.13059 ^[11]	0.20719 ^[9]	0.27557 ^[12]	0.18311 ^[6]	0.16899 ^[4]	0.19774 ^[8]	0.17488 ^[5]	0.18549 ^[7]	0.2428 ^[11]
	MRE	$\hat{\tau}$	0.19251 ^[11]	0.2076 ^[2]	0.23427 ^[7]	0.20904 ^[3]	0.22613 ^[6]	0.24024 ^[9]	0.21456 ^[4]	0.21495 ^[5]	0.2476 ^[12]	0.23455 ^[8]	0.24757 ^[10.5]	0.24757 ^[10.5]
		$\hat{\mu}$	0.08466 ^[2]	0.09237 ^[3]	0.10983 ^[9]	0.08354 ^[11]	0.10265 ^[8]	0.1193 ^[10]	0.09779 ^[6]	0.09412 ^[4]	0.10129 ^[7]	0.09622 ^[5]	0.30163 ^[12]	0.22269 ^[11]
$\sum Ranks$			9 ^[1]	15 ^[3]	53 ^[8]	13 ^[2]	46 ^[7]	63 ^[12]	29 ^[5]	27 ^[4]	58 ^[10]	41 ^[6]	59.5 ^[11]	54.5 ^[9]
275	BIAS	$\hat{\tau}$	0.04886 ^[11]	0.05221 ^[2]	0.05966 ^[7]	0.05241 ^[3]	0.05865 ^[6]	0.0666 ^[12]	0.05436 ^[5]	0.05313 ^[4]	0.06564 ^[11]	0.06159 ^[8]	0.06426 ^[10]	0.06398 ^[9]
		$\hat{\mu}$	0.24694 ^[11]	0.27267 ^[4]	0.32732 ^[10]	0.24887 ^[2]	0.31207 ^[8]	0.37171 ^[12]	0.27973 ^[5]	0.26875 ^[3]	0.31884 ^[9]	0.28918 ^[6]	0.30751 ^[7]	0.34525 ^[11]
	MSE	$\hat{\tau}$	0.00399 ^[11]	0.00461 ^[3]	0.00573 ^[7]	0.00452 ^[2]	0.0056 ^[6]	0.00744 ^[12]	0.0048 ^[5]	0.00471 ^[4]	0.00731 ^[11]	0.00637 ^[8]	0.00691 ^[9]	0.00696 ^[10]
		$\hat{\mu}$	0.09687 ^[2]	0.11649 ^[4]	0.16717 ^[10]	0.09491 ^[11]	0.1521 ^[8]	0.21973 ^[12]	0.12037 ^[5]	0.11327 ^[3]	0.1549 ^[9]	0.12837 ^[6]	0.14967 ^[7]	0.18415 ^[11]
	MRE	$\hat{\tau}$	0.16286 ^[11]	0.17405 ^[2]	0.19886 ^[7]	0.17471 ^[3]	0.1955 ^[6]	0.222 ^[12]	0.18119 ^[5]	0.17709 ^[4]	0.21878 ^[11]	0.2053 ^[8]	0.21421 ^[9.5]	0.21421 ^[9.5]
		$\hat{\mu}$	0.07056 ^[11]	0.07791 ^[4]	0.09352 ^[9]	0.07111 ^[2]	0.08916 ^[7]	0.1062 ^[10]	0.07992 ^[5]	0.07679 ^[3]	0.0911 ^[8]	0.08262 ^[6]	0.31089 ^[12]	0.21327 ^[11]
$\sum Ranks$			7 ^[1]	19 ^[3]	50 ^[8]	13 ^[2]	41 ^[6]	70 ^[12]	30 ^[5]	21 ^[4]	59 ^[10]	42 ^[7]	54.5 ^[9]	61.5 ^[11]
350	BIAS	$\hat{\tau}$	0.04352 ^[11]	0.04969 ^[5]	0.0506 ^[6]	0.04613 ^[2]	0.0539 ^[7]	0.05811 ^[11]	0.04751 ^[4]	0.04646 ^[3]	0.05461 ^[8]	0.05735 ^[10]	0.06191 ^[12]	0.0554 ^[9]
		$\hat{\mu}$	0.2255 ^[2]	0.25086 ^[5]	0.28363 ^[8]	0.22365 ^[11]	0.28392 ^[9]	0.325 ^[12]	0.24423 ^[4]	0.23665 ^[3]	0.26722 ^[7]	0.26576 ^[6]	0.2921 ^[10]	0.30329 ^[11]
	MSE	$\hat{\tau}$	0.00301 ^[11]	0.00417 ^[5]	0.00428 ^[6]	0.00368 ^[3]	0.00499 ^[7.5]	0.0056 ^[11]	0.00385 ^[4]	0.0034 ^[2]	0.00499 ^[7.5]	0.00546 ^[10]	0.00642 ^[12]	0.00506 ^[9]
		$\hat{\mu}$	0.0805 ^[2]	0.09893 ^[5]	0.1286 ^[8]	0.07972 ^[11]	0.12917 ^[9]	0.16517 ^[12]	0.0964 ^[4]	0.08838 ^[3]	0.11299 ^[7]	0.10712 ^[6]	0.12972 ^[10]	0.14171 ^[11]
	MRE	$\hat{\tau}$	0.14508 ^[11]	0.16564 ^[5]	0.16867 ^[6]	0.15378 ^[2]	0.17968 ^[7]	0.1937 ^[10]	0.15837 ^[4]	0.15486 ^[3]	0.18203 ^[8]	0.19115 ^[9]	0.20638 ^[11.5]	0.20638 ^[11.5]
		$\hat{\mu}$	0.06443 ^[2]	0.07167 ^[5]	0.08104 ^[8]	0.0639 ^[11]	0.08112 ^[9]	0.09286 ^[10]	0.06978 ^[4]	0.06761 ^[3]	0.07635 ^[7]	0.07593 ^[6]	0.30986 ^[12]	0.18466 ^[11]
$\sum Ranks$			9 ^[1]	30 ^[5]	42 ^[6]	10 ^[2]	48.5 ^[9]	66 ^[11]	24 ^[4]	17 ^[3]	44.5 ^[7]	47 ^[8]	67.5 ^[12]	62.5 ^[10]
450	BIAS	$\hat{\tau}$	0.03743 ^[11]	0.04214 ^[4]	0.04653 ^[7]	0.04149 ^[3]	0.04847 ^[9]	0.05128 ^[11]	0.04402 ^[5]	0.04142 ^[2]	0.05066 ^[10]	0.04603 ^[6]	0.05693 ^[12]	0.0472 ^[8]
		$\hat{\mu}$	0.19067 ^[11]	0.21679 ^[4]	0.25556 ^[8]	0.20299 ^[2]	0.2591 ^[9]	0.29261 ^[12]	0.2225 ^[6]	0.21271 ^[3]	0.2457 ^[7]	0.21943 ^[5]	0.26839 ^[11]	0.2654 ^[10]
	MSE	$\hat{\tau}$	0.00233 ^[11]	0.003 ^[4]	0.00348 ^[6]	0.00272 ^[2]	0.00377 ^[9]	0.0042 ^[10]	0.00312 ^[5]	0.00276 ^[3]	0.00428 ^[11]	0.00362 ^[7]	0.00538 ^[12]	0.00363 ^[8]
		$\hat{\mu}$	0.05901 ^[11]	0.07418 ^[4]	0.10044 ^[8]	0.06184 ^[2]	0.10419 ^[9]	0.13503 ^[12]	0.07865 ^[6]	0.07227 ^[3]	0.09547 ^[7]	0.07574 ^[5]	0.1111 ^[10]	0.11199 ^[11]
	MRE													

E. Method of least squares (LSE) (EM_5)

The method of least squares (LSE) is a statistical technique used to estimate the parameters of a mathematical model by minimizing the sum of the squares of the differences between the observed and predicted values. To estimate the parameters, we have to minimize the following equation:

$$V(x_i) = \sum_{i=1}^n \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2.$$

F. Method of right tail Anderson-Darling (EM_6)

The estimation method of right-tail Anderson-Darling involves determining the parameters of a statistical distribution, typically a CDF, by optimizing the right-tail version of the Anderson-Darling statistic,

$$R(x_i) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(x_{i:n}).$$

TABLE VII. Partial and overall ranks of all the methods of estimation of the proposed distribution by various values of model parameters.

Parameter	n	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}	EM_{11}	EM_{12}
$\tau = 0.9, \mu = 0.3$	25	3.5	8.0	10.0	1.0	7.0	12.0	2.0	5.0	6.0	3.5	11.0	9.0
	90	1.0	4.0	8.0	2.5	9.0	12.0	6.0	2.5	5.0	7.0	10.0	11.0
	150	1.0	6.0	9.0	3.0	10.0	12.0	5.0	2.0	7.0	4.0	8.0	11.0
	200	1.0	4.0	8.0	3.0	9.0	11.0	5.0	2.0	6.0	7.0	10.0	12.0
	275	1.0	6.0	7.0	3.0	11.0	12.0	4.0	2.0	9.0	5.0	8.0	10.0
	350	1.0	4.5	8.0	2.0	10.0	12.0	4.5	3.0	7.0	6.0	9.0	11.0
	450	1.0	4.0	7.0	2.0	9.0	12.0	5.5	3.0	8.0	5.5	10.0	11.0
	600	1.0	4.0	8.0	2.5	9.0	12.0	6.0	2.5	7.0	5.0	10.0	11.0
$\tau = 2.0, \mu = 0.8$	25	5.5	3.5	9.0	1.0	8.0	11.0	5.5	3.5	2.0	7.0	10.0	12.0
	90	2.5	5.0	10.0	4.0	9.0	12.0	7.0	1.0	2.5	6.0	8.0	11.0
	150	1.0	4.0	9.0	2.0	10.0	12.0	5.5	3.0	8.0	5.5	7.0	11.0
	200	1.0	4.0	9.0	2.5	10.0	12.0	7.0	2.5	5.0	6.0	8.0	11.0
	275	1.0	4.0	9.5	2.0	9.5	12.0	6.0	3.0	8.0	5.0	7.0	11.0
	350	1.0	4.0	9.5	3.0	9.5	12.0	6.0	2.0	7.0	5.0	8.0	11.0
	450	1.0	5.5	9.0	2.5	10.0	12.0	5.5	2.5	8.0	4.0	7.0	11.0
	600	1.0	4.0	10.0	2.0	9.0	12.0	6.0	3.0	8.0	5.0	7.0	11.0
$\tau = 1.5, \mu = 2.5$	25	3.0	4.0	9.0	2.0	6.5	12.0	11.0	5.0	1.0	6.5	8.0	10.0
	90	1.0	4.0	12.0	3.0	10.0	9.0	8.0	2.0	5.0	6.0	7.0	11.0
	150	1.0	4.0	8.0	3.0	9.0	12.0	7.0	2.0	5.0	6.0	10.0	11.0
	200	1.0	6.5	9.0	3.0	10.0	12.0	5.0	2.0	6.5	4.0	8.0	11.0
	275	1.0	5.0	9.0	2.0	10.0	12.0	6.0	3.0	7.0	4.0	8.0	11.0
	350	1.0	5.0	10.0	2.0	9.0	12.0	5.0	3.0	8.0	5.0	7.0	11.0
	450	1.0	6.0	8.0	2.0	10.0	12.0	4.0	3.0	7.0	5.0	9.0	11.0
	600	1.5	6.0	10.0	1.5	9.0	12.0	4.0	3.0	8.0	5.0	7.0	11.0
$\tau = 0.75, \mu = 1.5$	25	5.0	3.0	10.0	1.0	9.0	12.0	4.0	2.0	7.0	6.0	8.0	11.0
	90	1.0	4.0	8.0	2.5	10.0	12.0	6.0	2.5	9.0	5.0	7.0	11.0
	150	1.0	3.5	8.0	2.0	5.5	12.0	7.0	3.5	9.0	5.5	10.0	11.0
	200	1.0	4.0	8.0	3.0	9.0	12.0	6.0	2.0	10.0	5.0	7.0	11.0
	275	1.0	4.0	7.0	3.0	9.5	12.0	5.0	2.0	9.5	6.0	8.0	11.0
	350	1.0	4.0	7.0	2.0	11.0	12.0	5.0	3.0	9.0	6.0	8.0	10.0
	450	1.0	4.0	8.0	2.0	9.0	12.0	5.0	3.0	7.0	6.0	10.0	11.0
	600	1.0	6.0	7.0	2.5	9.0	12.0	4.0	2.5	8.0	5.0	10.0	11.0
$\tau = 2.5, \mu = 0.25$	25	4.0	6.0	11.0	3.0	5.0	7.0	10.0	1.0	2.0	8.0	9.0	12.0
	90	3.0	7.0	9.5	1.0	9.5	12.0	8.0	2.0	6.0	4.0	5.0	11.0
	150	1.0	5.0	10.0	2.0	9.0	12.0	7.0	3.0	4.0	6.0	8.0	11.0
	200	1.0	4.5	9.0	2.0	10.0	12.0	6.0	3.0	7.0	4.5	8.0	11.0
	275	1.0	5.0	9.0	2.0	10.0	12.0	6.0	3.0	7.0	4.0	8.0	11.0
	350	1.0	4.0	9.0	2.0	10.0	12.0	6.0	3.0	8.0	5.0	7.0	11.0
	450	1.0	4.5	10.0	2.0	9.0	12.0	6.0	3.0	8.0	4.5	7.0	11.0
	600	1.0	5.0	9.0	2.5	10.0	12.0	6.0	2.5	8.0	4.0	7.0	11.0
Σ ranks		60.0	188.5	354.5	91.0	367.0	470.0	233.5	106.5	269.5	212.5	329.0	438.0
Overall rank		1	4	9	2	10	12	6	3	7	5	8	11

G. Method of weighted least squares estimator (WLSE) (EM_7)

WLSE is a statistical method used to estimate the parameters of a probability model, where varying weights are assigned to each observation. To estimate the parameters, we have to minimize the following equation:

$$W(x_i) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2.$$

H. Method of left tail Anderson-Darling (EM_8)

The left-tail Anderson-Darling estimation method refers to the approach used to estimate the parameters of a distribution, particularly focusing on the left tail, based on the Anderson-Darling statistic. The Anderson-Darling statistic measures how well a given distribution fits a dataset. By minimizing the CDF of IUEPrD for ordered random variables, we can estimate the parameters of the suggested model,

$$L(x_i) = -\frac{3}{2}n + 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log F(x_{i:n}).$$

I. Minimum spacing absolute distance (EM_9)

If we have an ordered sample denoted as $X_{1:n}, \dots, X_{n:n}$ from IUEPrD, the minimum spacing absolute distance estimates can be derived by minimizing the following equation:

$$\zeta(x_i) = \sum_{i=1}^{n+1} \left| \hat{\omega}_i - \frac{1}{n+1} \right|.$$

J. Minimum spacing absolute-log distance (EM_{10})

Using this estimation method, we can estimate the model parameters by minimizing the following expression:

$$Y_2(x_i) = \sum_{i=1}^{n+1} \left| \log \hat{\omega}_i - \log \frac{1}{n+1} \right|.$$

K. Anderson-Darling method 2LT left-tail second-order (EM_{11})

The parameters of IUEPrD can be derived using the left-tail second-order estimation using the Anderson-Darling method 2LT, which is achieved by minimizing the following function:

$$LTS = 2 \sum_{i=1}^n \log F(x_i) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{F(x_i)}.$$

L. Kolmogorov method (EM_{12})

The Kolmogorov estimators for the parameters of the IUEPrD distribution can be obtained by optimizing the following expression:

$$KM = \max_{1 \leq i \leq n} \left[\frac{i}{n} - F(x_i), F(x_i) - \frac{i-1}{n} \right].$$

V. NUMERICAL SIMULATION

In this section of our paper, we aim to assess the efficacy of different estimation methods in determining the distribution of parameters associated with the IUEPrD. To achieve this goal, we adopt a simulation-based approach, systematically evaluating the performance of these estimation techniques. Our investigation spans various sample sizes, denoted as $n = 25, 90, 150, 200, 275, 350, 450, 600$, and explores diverse parameter configurations. We generate 5000 random samples drawn from the IUEPrD to facilitate our analysis. We then proceed to evaluate the performance of these estimation methods by computing three key statistical metrics: the average absolute bias (BIAS), the Mean Square Error (MSE), and the Mean Relative Estimate (MRE). These computations use the statistical software package `optim` in R.¹⁸

By systematically varying both sample size and parameter values, we provide valuable insights into the performance of these methods across different scenarios. These insights are particularly relevant for researchers and practitioners seeking reliable and robust estimation techniques. The expressions for BIAS, MSE, and MRE are given by

$$\begin{aligned} \text{BIAS} &= \frac{1}{M} \sum_{i=1}^M |\hat{\Phi}_i - \Phi_i|, & \text{MSE} &= \frac{1}{M} \sum_{i=1}^M (\hat{\Phi}_i - \Phi_i)^2, \\ \text{MRE} &= \frac{1}{M} \sum_{i=1}^M \frac{|\hat{\Phi}_i - \Phi_i|}{\Phi_i}, \end{aligned}$$

where Φ_i represents τ and μ .

Tables I–VI present our simulation results. These tables display the computed values of the three fundamental performance metrics obtained using 12 distinct estimation methods. Notably, the data in these tables consistently exhibit relatively small values across various parameter combinations. This observation underscores the tendency of the estimators to produce results with minimal deviations from the true values. Importantly, this trend reflects the consistency in the estimators, wherein the magnitudes of these metrics decrease as the sample size increases. This pattern is consistently observed across all experimental scenarios. Based on the information provided in Table VII, we infer that the most effective methods for estimating our proposed model parameters are EM_6 and EM_{12} , as they consistently achieve high rankings in the simulation tables.

VI. REAL DATA ANALYSIS

In this section, we provide two real data examples to show the usefulness of the IUEPrD in modeling real-world data. In this regard, we consider some selection criteria, such as $-2 \times \log(\ell)$,

TABLE VIII. List of the lifetime distribution to modeling datasets.

$f_W(\alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{x}{\beta} \right)^{\alpha-1} \exp \left(- \left(\frac{x}{\beta} \right)^{\alpha} \right),$	$\alpha, \beta > 0$
$f_{EP}(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} e^{\left(\frac{x}{\alpha} \right)^{\beta}} \left[1 - e^{\left(\frac{x}{\alpha} \right)^{\beta}} \right],$	$\alpha, \beta > 0$
$f_{EW}(\alpha, \beta, \gamma) = \alpha \beta \gamma x^{\gamma-1} \exp(-\alpha x^{\gamma}) \{1 - \exp(-\alpha x^{\gamma})\}^{\beta-1},$	$\alpha, \beta, \gamma > 0$
$f_{EE}(\alpha, \beta) = \alpha \beta \exp(-\beta x) (1 - \exp(-\beta x))^{\alpha-1},$	$\alpha, \beta > 0$

TABLE IX. MLEs and SEs of the parameters of the fitted distributions for the flood charges data.

Distribution	$\hat{\tau}$	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$SE(\hat{\tau})$	$SE(\hat{\mu})$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$	$SE(\hat{\gamma})$
IUEPrD	0.1288	36.4482	...	...	...	0.0614	5.8516	...	...	...
Weibull	...	...	1.7724	58.3858	...	...	...	0.1777	5.0597	...
EP	...	...	88.6411	1.0907	...	...	...	7.4340	0.1146	...
EE	...	...	5.8497	0.0480	...	...	...	1.7322	0.0070	...
EW	...	...	0.7991	0.4819	93.7547	...	...	0.5435	0.1188	112.1037

TABLE X. Selection criteria for the flood charges data.

	$-2 \log \ell$	AIC	BIC	CAIC	HQIC
IUEPrD	434.4560	438.4560	442.1984	438.7226	439.8702
Weibull	451.4130	455.4130	459.1554	455.6797	456.8272
EP	464.9439	468.9439	472.6863	469.2106	470.3582
EE	438.8942	442.8942	446.6366	443.1609	444.3085
EW	432.5116	438.5116	444.1252	439.0571	440.6330

Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan–Quinn information criterion (HQIC), to compare the fits of fitted distributions. Two real-life datasets are fitted to IUEPrD, Weibull, Exponential Power (EP),¹⁹ exponentiated

exponential (EE),²⁰ and exponentiated Weibull (EW)²¹ distributions. The PDFs of the fitted distributions are given in Table VIII.

A. Flood charges data

The first dataset consists of 48 observations related to the annual flood discharges of the North Saskatchewan River at Edmonton obtained by Ref. 22.

The flood charges data are given by 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, and 185.560.

The MLEs and the corresponding standard errors (SEs) of the parameters of the fitted models are given in Table IX, while Table X

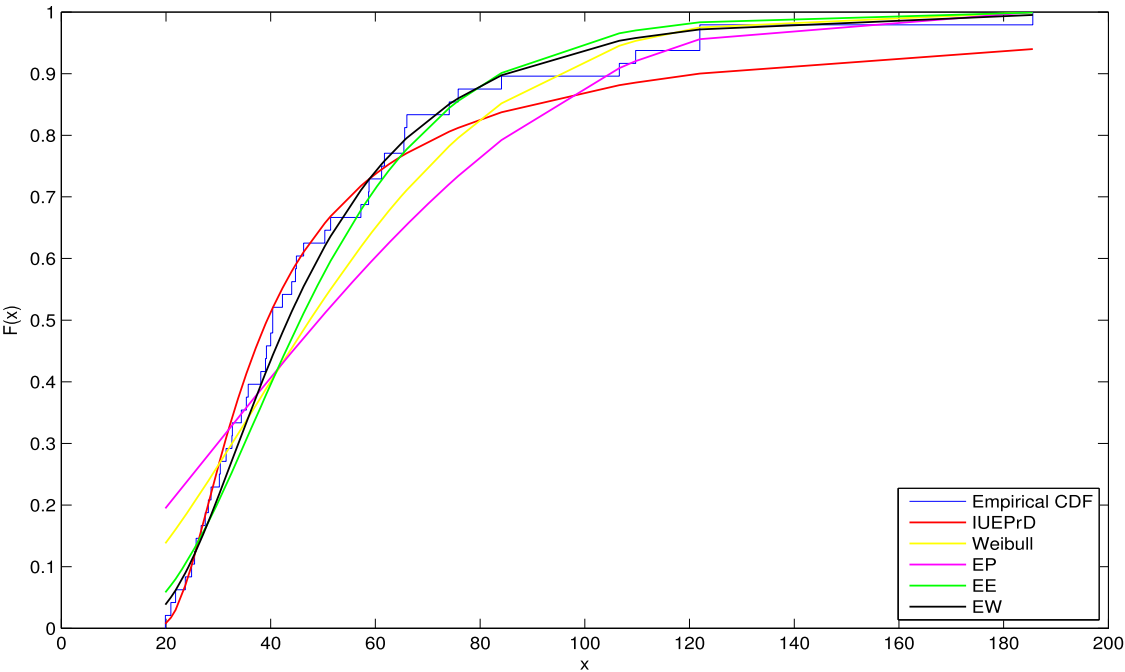


FIG. 2. Fitted CDFs for the flood charges dataset.

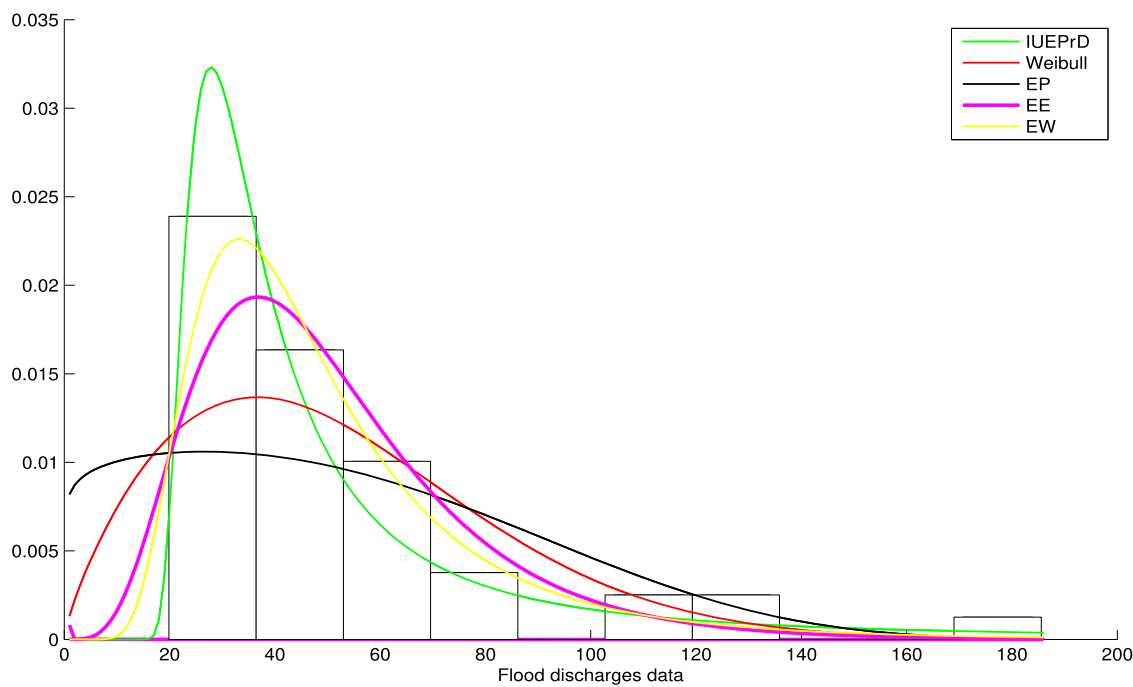


FIG. 3. Fitted PDFs for the flood charges dataset.

presents the selection criteria for the flood charges data. Furthermore, Figs. 2 and 3 illustrate the fitted CDFs and PDFs for the flood charges data, respectively.

From Table X, we conclude that the IUEPrD distribution is the best-fitted model for the flood charges dataset according to all comparison criteria.

B. Vehicle pass time data

The second dataset is provided by Ref. 23 regarding the length of intervals between times at which vehicles pass a point on a road. The second dataset is 2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3, 3, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5, 5.6, 5.7, 6, 6, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8, 8.3, 8.8, 9.3, 9.4, 9.5, 10.1, 11, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42,

TABLE XII. Selection criteria for the vehicle pass time data.

	$-2\log \ell$	AIC	BIC	CAIC	HQIC
IUEPrD	665.6352	669.6352	674.4729	669.7852	671.5787
Weibull	677.2643	681.2643	686.1020	681.4143	683.2078
EP	688.6168	692.6168	697.4545	692.7668	694.5603
EE	676.7629	680.7629	685.6005	680.9129	682.7064
EW	664.0233	670.0233	677.2798	670.3271	672.9386

44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, and 119.8. Table XI provides the MLEs and the corresponding standard errors of the parameters of the fitted distributions. Table XII provides the vehicle pass time dataset selection criteria. In addition, Figs. 4

TABLE XI. MLEs and SEs of the parameters of the fitted distributions for the vehicle pass time data.

Distribution	$\hat{\tau}$	$\hat{\mu}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$SE(\hat{\tau})$	$SE(\hat{\mu})$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$	$SE(\hat{\gamma})$
IUEPrD	8.1589	0.4322	...	...	...	11.0385	0.5249	...	...	...
Weibull	...	...	0.9962	21.7171	...	...	...	0.0816	2.5390	...
EP	...	...	42.6472	0.6810	...	...	...	4.2965	0.0596	...
EE	...	...	1.1132	0.0493	...	...	...	0.1668	0.0070	...
EW	...	...	2.8189	0.2263	97.0826	...	...	1.8362	0.0931	198.8596

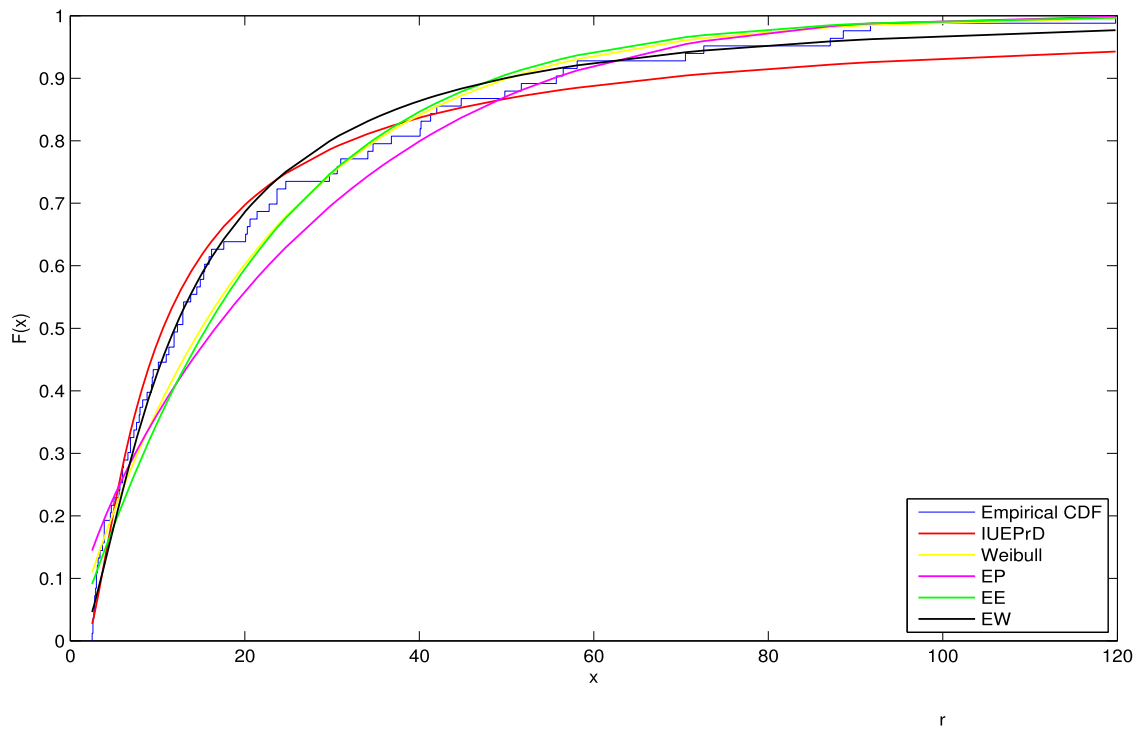


FIG. 4. Fitted CDFs for the vehicle pass time dataset.

and 5 illustrate the fitted CDFs and PDFs for the vehicle pass time dataset.

Table XII shows that the IUEPrD distribution is the best-fitted model for the vehicle pass time dataset according to all

comparison criteria except $-2\log \ell$. As a result of the goodness-of-fit test for the flood charges and vehicle pass time datasets, the IUEPrD distribution is a worthy alternative to Weibull and other competitors.

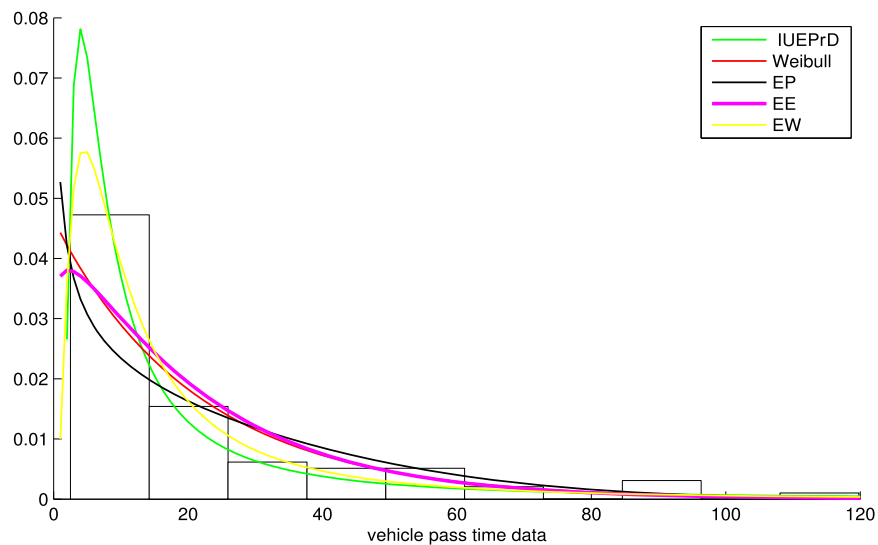


FIG. 5. Fitted PDFs for the vehicle pass time dataset.

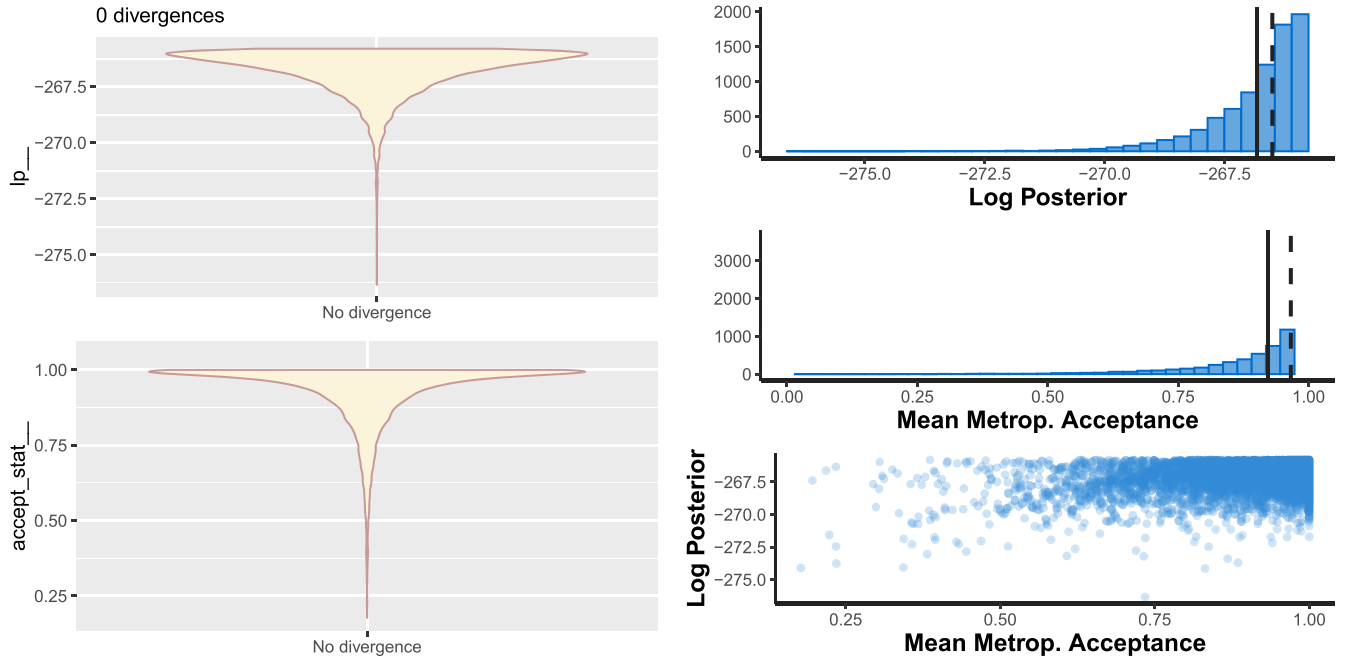


FIG. 6. Sampling information.

VII. BAYESIAN ANALYSIS OF IUEPrD

In this section, we analyze and estimate the parameters of the IUEPrD using the Bayesian approach. To perform the Bayesian analysis, we require prior distributions of the parameters and likelihood functions of the IUEPrD, which are defined as follows: we assume that the prior distribution of the parameters μ and τ follows a gamma prior, represented as

$$p(\mu) = \frac{b_1^{a_1}}{\Gamma(a_1)} \mu^{a_1-1} e^{-b_1 \mu}, \mu > 0, (a_1, b_1) > 0,$$

$$p(\tau) = \frac{b_2^{a_2}}{\Gamma(a_2)} \tau^{a_2-1} e^{-b_2 \tau}, \tau > 0, (a_2, b_2) > 0.$$

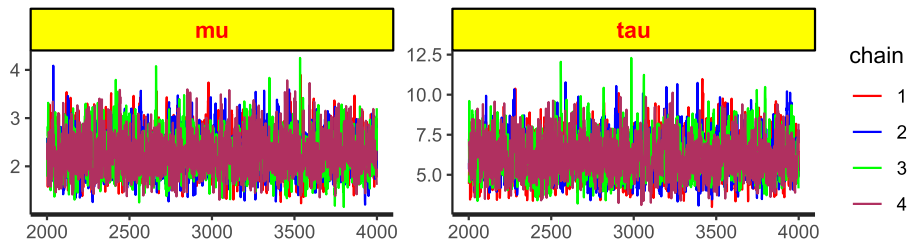
Furthermore, the likelihood function for the parameters μ and τ and the observed data $\underline{x} = (x_1 \dots x_n)$ can be obtained as

$$L(\mu, \tau | \underline{x}) = (2\mu\tau)^n \prod_{i=1}^n \frac{\left(\frac{x_i+1}{x_i-1}\right)^\mu e^{\tau-\tau\left(\frac{x_i+1}{x_i-1}\right)^\mu}}{x_i^2 - 1}.$$

Now, we can define the posterior distribution by using the priors and likelihood function as

$$\begin{aligned} p(\mu, \tau | \underline{x}) &= L(\mu, \tau | \underline{x}) \cdot p(\mu)p(\tau) \\ &\propto (2\mu\tau)^n \mu^{a_1-1} \tau^{a_2-1} e^{-b_1 \mu - b_2 \tau} \\ &\quad \times \prod_{i=1}^n \frac{\left(\frac{x_i+1}{x_i-1}\right)^\mu e^{\tau-\tau\left(\frac{x_i+1}{x_i-1}\right)^\mu}}{x_i^2 - 1}. \end{aligned}$$

For the numerical analysis of the posterior distribution, we have used the flood charges data presented in Sec. VI A, and we have considered the hyperparameters as $\mu \sim \Gamma(6, 10)$ and $\tau \sim \Gamma(1.5, 2.5)$.

FIG. 7. Trace plot for the parameters μ and τ .

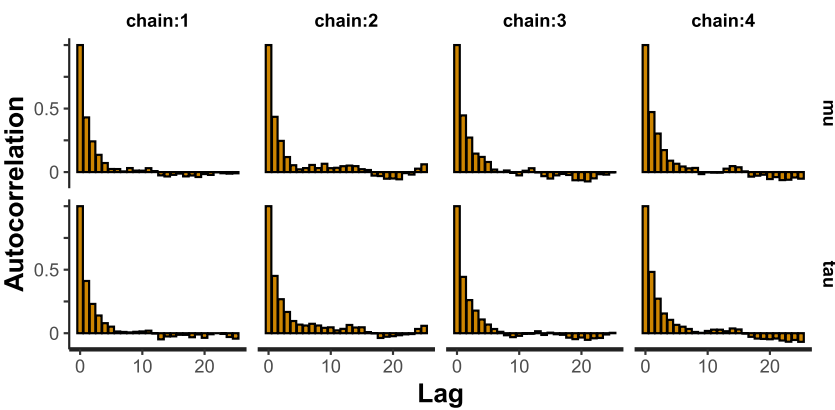


FIG. 8. Autocorrelation plot for the parameters μ and τ .

TABLE XIII. Posterior summary statistics.

Parameter	Mean	se_mean	sd	2.50%	25%	50%	75%	97.50%	n_eff	Rhat
mu	2.2516	0.0082	0.4026	1.5477	1.9626	2.2219	2.5110	3.1163	2418.5660	1.0026
tau	6.0742	0.0247	1.2049	4.0186	5.2053	5.9781	6.8277	8.6733	2362.5480	1.0024
lp__	-266.8240	0.0184	1.0119	-269.5110	-267.2130	-266.5050	-266.1060	-265.8360	3000.9730	1.0014

A. Convergence analysis

The assessment of convergence in Stan for Bayesian inference is essential to ensure that the MCMC sampling process adequately

explores the posterior distribution. Graphical visualization of convergence diagnostics aids in this assessment.²⁴ Figure 6 presents information regarding the sampling process, indicating the absence of divergent transitions, a positive indicator of convergence. In

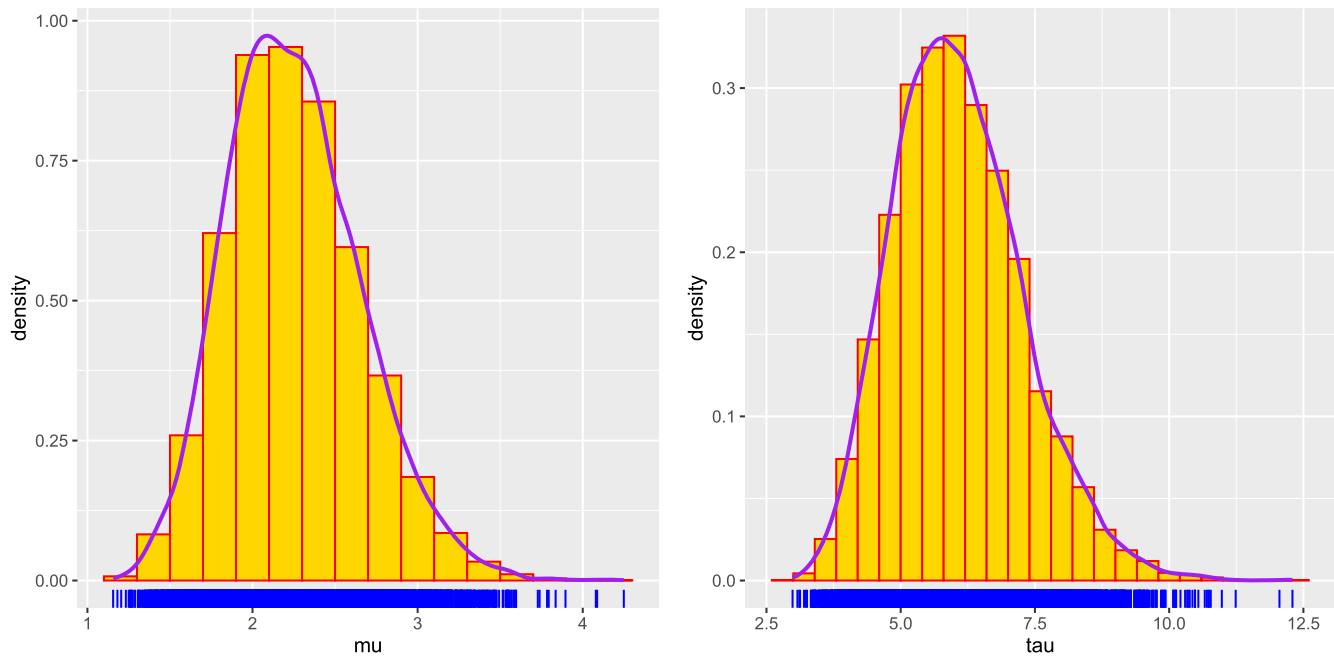


FIG. 9. Histograms of posterior estimates of parameters μ and τ .

TABLE XIV. Credible and HPD intervals at 95% confidence level.

Parameter	CI	HPD
μ	(1.55 3.12)	(1.52 3.08)
τ	(4.02 8.67)	(3.87 8.48)
lp_{-}	(-269.51 -265.84)	(-268.86 -265.81)

Fig. 7, we observe trace plots for parameters μ and τ . These plots demonstrate that the chains are well-mixed and have converged, further supporting the adequacy of the sampling process. Moreover, Fig. 8 shows the autocorrelation plots for different chains, suggesting that the samples are independent and uncorrelated. This independence is crucial for ensuring the validity of the sampled posterior distribution. Hence, the convergence analysis in Stan, as illustrated through graphical diagnostics, provides confidence that the MCMC sampling process has effectively explored the posterior distribution and reached convergence.

B. Posterior sample analysis

After conducting the MCMC algorithm in Stan, we acquired a substantial number of samples (i.e., 16 000, with 8000 utilized for warm-up) from the posterior distribution of parameters. These samples facilitated parameter estimation and uncertainty quantification through various summary statistics, including mean, median (50%), Monte Carlo standard error (se_mean), standard deviation (SD), effective sample size (n_eff), and Gelman–Rubin statistics (Rhat), which are listed in Table XIII. In addition, we present a graphical representation of histograms of posterior estimates of the parameters μ and τ in Fig. 9.

Moreover, to further characterize the uncertainty, we computed 95% credible intervals (CIs) and highest posterior density (HPD) intervals, which are presented in Table XIV. These intervals offer insights into the range of plausible parameter values based on the observed data and the posterior distribution. These analyses enable us to draw conclusions and make predictions based on the model's performance and its compatibility with the observed data.

VIII. CONCLUSION

Utilizing the inverse transformation approach, we have introduced a novel distribution termed the Inverse Unit Exponential Distribution (IUEPrD). This distribution exhibits a reverse-J, increasing, or inverted bathtub-shaped hazard rate function. We have delineated key statistical properties characterizing this distribution through a comprehensive analysis. The associated parameters were estimated using 12 methods, and a Monte Carlo simulation was conducted to assess the estimation procedure. Our findings indicate that biases and Mean Squared Errors (MSEs) diminish with increasing sample sizes, even for small samples. To validate and apply the IUEPrD, we leveraged two real datasets using the classical approach. Our analysis revealed that the proposed model outperforms existing models documented in the literature.

Furthermore, we subjected the suggested model to scrutiny under the Bayesian framework. Employing Markov Chain Monte Carlo (MCMC) with Hamiltonian Monte Carlo (HMC) under the No U-Turn Sampling (NUTS) scheme, we generated 8000 real

samples. Convergence diagnostics revealed that the chains exhibit convergence and the samples demonstrate independence. We subsequently estimated the parameters of the IUEPrD using posterior samples and computed credible intervals (CIs) and Highest Posterior Density (HPD) intervals. Thus, our investigation demonstrates the applicability of the suggested model to real data under both classical and Bayesian methodologies.

ACKNOWLEDGMENTS

This research is supported by researchers supporting project number (RSPD2024R548), King Saud University, Riyadh, Saudi Arabia.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

N.A., C.T., L.P.S., A.K., W.M., and A.M.G. have contributed equally to this work.

Najwan Alsadat: Methodology (equal); Software (equal). **Caner Taniş:** Conceptualization (equal); Data curation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Laxmi Prasad Sapkota:** Conceptualization (equal); Data curation (equal); Investigation (equal); Software (equal). **Anoop Kumar:** Data curation (equal); Visualization (equal); Writing – review & editing (equal). **Waleed Marzouk:** Conceptualization (equal); Project administration (equal); Software (equal); Validation (equal). **Ahmed M. Gemeay:** Project administration (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

- ¹K. V. P. Barco, J. Mazucheli, and V. Janeiro, "The inverse power Lindley distribution," *Commun. Stat. - Simul. Comput.* **46**(8), 6308–6323 (2017).
- ²R. Jan, T. Jan, and P. B. Ahmad, "Exponentiated inverse power Lindley distribution and its applications," *arXiv:1808.07410* (2018).
- ³A. S. Hassan and M. Abd-Allah, "On the inverse power Lomax distribution," *Ann. Data Sci.* **6**, 259–278 (2019).
- ⁴M. M. Al Sobhi, "The inverse-power logistic-exponential distribution: Properties, estimation methods, and application to insurance data," *Mathematics* **8**(11), 2060 (2020).
- ⁵C. Chesneau and V. Agiwal, "Statistical theory and practice of the inverse power Muth distribution," *J. Comput. Math. Data Sci.* **1**, 100004 (2021).
- ⁶H. Al-Kzzaz and M. Abd El-Monsef, "Inverse power Maxwell distribution: Statistical properties, estimation and application," *J. Appl. Stat.* **49**(9), 2287–2306 (2022).

- ⁷E. Yıldırım, E. S. Ilikkan, A. M. Gemeay, N. Makumi, M. Bakr, and O. S. Balogun, "Power unit Burr-XII distribution: Statistical inference with applications," *AIP Adv.* **13**(10), 105107 (2023).
- ⁸N. Alsadat, M. Elgarhy, K. Karakaya, A. M. Gemeay, C. Chesneau, and M. Abd El-Raouf, "Inverse unit Teissier distribution: Theory and practical examples," *Axioms* **12**(5), 502 (2023).
- ⁹A. M. Alsahangiti, A. M. Gemeay, L. P. Sapkota, M. E. Bakr, S. Benchiha, O. S. Balogun, E. Hossam, and M. S. Shama, "Inverse power generalized Maxwell distribution with applications in industry," *Research Square* preprint (2023).
- ¹⁰L. P. Sapkota and V. Kumar, "Applications and some characteristics of inverse power Cauchy distribution," *Reliab.: Theory Appl.* **18**(1), 301–315 (2023).
- ¹¹Stan Development Team, RStan: The R interface to Stan. R package version 2.32.5 (2024a).
- ¹²B. Carpenter, A. Gelman, M. D. Hoffman, D. Lee, B. Goodrich, M. Betancourt, M. A. Brubaker, J. Guo, P. Li, and A. Riddell, "Stan: A probabilistic programming language," *J. Stat. Software* **76**, 1–32 (2017).
- ¹³Stan Development Team, The Stan Core Library. Version 2.31.0 (2024b).
- ¹⁴M. H. AbuJarad and A. A. Khan, "Bayesian survival analysis of Topp–Leone generalized family with Stan," *Int. J. Stat. Appl.* **8**(5), 274–290 (2018).
- ¹⁵L. P. Sapkota, "A Bayesian analysis and estimation of Weibull inverse Rayleigh distribution using HMC method," *Nepal J. Math. Sci.* **3**(2), 39–58 (2022).
- ¹⁶L. P. Sapkota, A. M. Alsahangiti, V. Kumar, A. M. Gemeay, M. Bakr, O. S. Balogun, and A. H. Muse, "Arc-tangent exponential distribution with applications to weather and chemical data under classical and Bayesian approach," *IEEE Access* **11**, 115462–115476 (2023).
- ¹⁷H. S. Bakouch, T. Hussain, M. Tošić, V. S. Stojanović, and N. Qarmalah, "Unit exponential probability distribution: Characterization and applications in environmental and engineering data modeling," *Mathematics* **11**(19), 4207 (2023).
- ¹⁸R Core Team, *R: A Language and Environment for Statistical Computing* (R Foundation for Statistical Computing, Vienna, Austria, 2023).
- ¹⁹R. M. Smith and L. J. Bain, "An exponential power life-testing distribution," *Commun. Stat. - Simul. Comput.* **4**(5), 469–481 (1975).
- ²⁰R. D. Gupta and D. Kundu, "Exponentiated exponential family: An alternative to gamma and Weibull distributions," *Biom. J.* **43**(1), 117–130 (2001).
- ²¹M. Pal, M. M. Ali, and J. Woo, "Exponentiated Weibull distribution," *Statistica* **66**(2), 139–147 (2006).
- ²²M. Van Montfort, "On testing that the distribution of extremes is of type I when type II is the alternative," *J. Hydrol.* **11**(4), 421–427 (1970).
- ²³B. Jorgensen, *Statistical Properties of the Generalized Inverse Gaussian Distribution* (Springer Science & Business Media, 2012), Vol. 9.
- ²⁴G. Andrew, J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari, and D. B. Rubin, *Bayesian Data Analysis* (Chapman and Hall/CRC, 2013).