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RESEARCH

Comparative Analysis of the Singularly Perturbed Generalized Burgers-Huxley Problem via Approximate Lie Symmetry and Exponentially Fitted Finite Element Method

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Abstract

Singularly perturbed generalized Burgers-Huxley equation (SPGBHE) models nonlinear wave phenomena in fluid dynamics, combustion, and biological systems. This study addresses the solutions of SPGBHE through approximate Lie symmetry analysis (LSA) and finite element method (FEM), with rigorous comparison validating solutions obtained through both methodologies. The method of approximate symmetry, which involves expanding the infinitesimal generator in a perturbation series is implemented to solve the governing equation, yielding approximate infinitesimal generators essential for constructing the optimal system of Lie sub-algebra. A set of group invariant solutions is determined for each reduction derived from corresponding sub-algebras mentioned in the system. On the second part, exponentially fitted finite element method (EF-FEM) along with the explicit Euler scheme is implemented using a piecewise uniform Shishkin mesh to examine the equation from a numerical perspective. Stability and uniform convergence of outlined approach is discussed to provide credibility to the numerical scheme. Moreover, solutions derived through each technique are authenticated by firm comparison following a detailed error analysis, including error table and comparison plots. Visual representations of solution profiles for specific outcomes of LSA are also presented to see the impact of the singular perturbation parameter (ϵ) alongside other parameters.

Keywords Lie symmetry · Optimal system · Tanh-coth function method · Analytical solutions · Approximate solutions · Finite element method · Shishkin mesh · Explicit Euler method

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1 Introduction

Singularly perturbed partial differential equations (SPPDEs) represent a class among partial differential equations (PDEs) in which there is a presence of a small-scale positive parameter generally symbolized by ϵ (singular perturbation parameter); $0 < \epsilon \ll 1$ induces different behaviors within distinct domain segments, which enables distinguishing between fast and slow processes within the dynamics of the system. SPPDEs find extensive applications in chemical reactor theory [1], gas porous electrodes theory [2], optimal control problems [3], and population dynamics [4], among many other domains [5]. Numerous researchers have been captivated by quantitative and qualitative analysis of the aforementioned equations. The two primary approaches employed for achieving solutions to singularly perturbed nonlinear differential equations are numerical analysis [6] and asymptotic analysis [7]. The main challenge in acquiring solutions arise from the existence of boundary layers in the solution as ϵ approaches small values, necessitating a precise approximation to enhance the reliability of the approximate solution.

The problem under consideration is one such SPPDE, namely singularly perturbed generalized Burgers-Huxley equation. Many authors have worked on the solution methodologies for generalized Burgers-Huxley equation (BHE). Hashim et al. [8] employed the Adomian decomposition method for BHE and achieved analytical solution as a convergent power series. Deng [9] studied the first integral method to find traveling wave solutions. Painlevé analysis [10], implicit exponential finite difference method [11] among others [12–14] are some more methods for the same. However, these were methodologies adopted for the case when $\epsilon = 1$. Few authors have worked on the equation involving the perturbation parameter, which, when tended to 0, exhibits boundary layers. Most of conventional approaches fail to capture this effect. Kumar et al. [15] carried out a numerical study and utilized a three-step Taylor-Galerkin approach to capture the boundary layers. Daba and Duressa [16] addressed SPGBHE numerically by conducting linearization followed by the utilization of Euler method and finite difference method (FDM). Derzie et al. [17] formulated a numerical scheme for this equation by using Crank-Nicolson scheme and a fitted operator upwind FDM. They were successful in achieving convergence of order two in time and one with respect to space. The variational iteration method by Kamboj et al. [18] was used to obtain the analytical solution via iteration.

The approaches opted for solving the generalized SPPDE in this study are symmetry analysis and finite element method. Lie theory of continuous transformations, first introduced by Marius Sophus Lie, later brought to prominence by Ovsiannikov [19] provides a comprehensive and potent framework for addressing differential equations (refer [20–23]). However, any small perturbation of an equation tends to destroy certain crucial symmetries, limiting the efficacy of Lie group method in solving differential equations. In contrast, perturbation techniques are effectively employed to study this type of differential equation. Hence, combining the Lie group method with perturbation analysis is advantageous for establishing approximate symmetries. In this context, two distinct approximate symmetry approaches have been formulated. One of them is presented by Baikov, Gazizov, and Ibragimov [24], in 1988, which is a transformation technique (BGI method) based on expanding the symmetry generator within perturbation series with respect to perturbation parameter. In contrast, Fushchich and Shtelen in 1989 developed the FS method [25] by combining symmetry group method with the perturbation techniques. This involves expressing the dependent variable as Taylor series in ϵ. It is followed by substituting the governing equation with set of equations correspond to the terms at various powers of ϵ. Note that these are the primary approaches that are widely used, and we stick to the former approach for obtaining exact and approximate solutions of the equation under consideration.

Finite element method (FEM) serves as a effective computational approach broadly used to tackle complex engineering and physical problems. In the area of mathematics, it is recognized as a systematically efficient numerical technique which aims to approximate the solution of partial differential equations (PDEs) over a known domain Ω. The classical solution of a differential equation, say u, must be sufficiently differentiable; however, it may not be achievable in some scenarios. FEM serves it's advantage in the sense that it relaxes the smoothness requirement over the solution. Developed in the mid-20th century, FEM has revolutionized the fields of fluid dynamics [26], structural analysis [27], heat transfer [28], electromagnetism [29] including [30, 31] and many others by providing a flexible and accurate approach to modeling and solving differential equations. Due to its versatility and systematic procedure, numerous researchers and mathematicians have been using it to address a diverse range of problems. Here, the primary objective is to explore the utility of approximate symmetry analysis and exponentially fitted FEM for analytically and numerically solving the singularly perturbed form of the generalized Burgers-Huxley problem and to authenticate the solutions derived via both approaches. So far, to the author's awareness, the approximate solutions obtained via LSA have not been validated with any numerical scheme for SPGBHE. To navigate the key sections, this study is arranged as follows:

- ≏ Section 1: Introduction of the problem; Historical analysis by various researchers.
- ≏ Section 2: The singularly perturbed equation under consideration.
- ≏ Section 3: Preliminary aspects of the method for approximate symmetry analysis on the SPGBHE; Acquisition of optimal system of Lie subalgebras through point symmetries.
- ≏ Section 4: Reductions of the main equation; Application of the tanh-coth method; Solution(s) of the reduced equations.
- ≏ Section 5: Exponentially fitted finite element scheme: Time semi-discretization, Spatial discretization; Quasilinearization process.
- ≏ Section 6: Stability and uniform convergence of the proposed EF-FEM.
- ≏ Section 7: FE and LS solution: Comparative analysis via error table and solution plots.
- ≏ Section 8: Visualization of exact solutions obtained via LSA.
- ≏ Section 9: Conclusion of the work.

2 Singularly Perturbed Generalized Burgers-Huxley Equation

Singularly perturbed generalized Burgers-Huxley equation (SPGBHE) signifies a nonlinear time-dependent problem which represents wave phenomena and is presented as:

$$u_t + \alpha u^{\delta} u_x - \epsilon u_{xx} = \beta u(1 - u^{\delta})(u^{\delta} - \gamma). \tag{2.1}$$

Here, x and t represent independent variables, while u denotes dependent variable. Parameters α, β, γ, δ and ϵ are subject to the conditions. α, β are non-negative, γ lies in the interval $(0, 1)$, δ can take the values $1,\ 2,\ 3$, and ϵ is such that $0 < \epsilon \ll 1$. A particular case of (2.1) is obtained when ϵ is set to 1. It is known as generalized BHE and is viewed as a framework that defines interplay among convection effects, reaction mechanisms and diffusion transports. Generalized BHE encompasses various well-established evolution equations as specific instances - with $\alpha = 0$, $\delta = 1$, it is the Huxley equation; with $\delta = 1$, $\beta = 0$, it is the Burgers equation; with $\delta = 1$, $\alpha \neq 0$, $\beta \neq 0$, it represents Burgers-Huxley equation; with $\delta = 1$, $\alpha = 0$, it is the Fitz-Hugh-Nagumo equation; with

$\delta = 1, \alpha = 0, \gamma = -1$, is the Newell-Whitehead problem. Our target equation constitute (2.1). For further computations, we take the transformation

$$u = v^{\frac{1}{\delta}}, \tag{2.2}$$

causing the main equation to transform into

$$v_t + \alpha v\,(v_x) - \delta\beta v^2 + \delta\beta v\gamma + \delta\beta v^3 - \delta\beta v^2\gamma + \epsilon\left(-v_{xx} - \left(\frac{1}{\delta} - 1\right)(v_x)^2\, v^{-1}\right) = 0. \tag{2.3}$$

Our subsequent analysis will focus on finding solutions to (2.3), which, after applying transformation in (2.2) yields the solution to the governing (2.1).

3 Symmetry Analysis of the Transformed SPGBHE

This section introduces the approximate symmetry method [24] for perturbed differential equations. Followed by the application of stated method on the governing equation, we are able to find optimal system of one dimensional subalgebras for (2.1) using adjoint representation.

3.1 Preliminaries

Consider $\mathbb{G}$ as the one-parameter (ρ) approximate transformation group

$$\begin{aligned}
\bar{x} &= x + \rho\varsigma(x, t, v, \epsilon) + O(\rho^2),\\
\bar{t} &= t + \rho\varpi(x, t, v, \epsilon) + O(\rho^2),\\
\bar{v} &= v + \rho\varphi(x, t, v, \epsilon) + O(\rho^2),
\end{aligned} \tag{3.1}$$

whereby the infinitesimals ς, ϖ, and φ are

$$\begin{aligned}
\varsigma(x, t, v, \epsilon) &= \sum_{i=0}^{p} \epsilon^i \varsigma_i(x, t, v, \epsilon),\\
\varpi(x, t, v, \epsilon) &= \sum_{i=0}^{p} \epsilon^i \varpi_i(x, t, v, \epsilon),\\
\varphi(x, t, v, \epsilon) &= \sum_{i=0}^{p} \epsilon^i \varphi_i(x, t, v, \epsilon).
\end{aligned}$$

ρ is the group parameter, with ϵ representing the perturbation parameter. For $\mathbb{G}$, the vector field corresponding to (3.1) is

$$\mathfrak{B} = \varsigma(x, t, v, \epsilon)\partial_x + \varpi(x, t, v, \epsilon)\partial_t + \varphi(x, t, v, \epsilon)\partial_v.$$

Considering $\mathbb{G}$ in the first order of precision. Let infinitesimal generator be

$$\mathfrak{B} = \mathfrak{B}_0 + \epsilon\mathfrak{B}_1, \tag{3.2}$$

where

$$\begin{aligned}
\mathfrak{B}_0 &= \varsigma_0(x, t, v, \epsilon)\partial_x + \varpi_0(x, t, v, \epsilon)\partial_t + \varphi_0(x, t, v, \epsilon)\partial_v,\\
\mathfrak{B}_1 &= \varsigma_1(x, t, v, \epsilon)\partial_x + \varpi_1(x, t, v, \epsilon)\partial_t + \varphi_1(x, t, v, \epsilon)\partial_v.
\end{aligned}$$

Theorem 1 [32] $\mathbb{G}$ *identifies as approximate symmetry group with respect to the perturbed PDE ($\Xi = 0$) if invariance condition*

$$[\mathfrak{B}^2(\Xi(v,\epsilon))]\big|_{\Xi\approx 0} = o(\epsilon), \tag{3.3}$$

is satisfied in case of each infinitesimal generator within Lie algebra associated with $\mathbb{G}$, alongside $\mathfrak{B}$ defined in (3.2). Note that the converse also holds.

Remark 3.1 The aforementioned theorem is stated for a perturbed differential equation

$$\Xi(x,t,v,v_x,v_{xx},...,\epsilon) \equiv \Xi_0(x,t,v,v_x,v_{xx},...) + \epsilon\Xi_1(x,t,v,v_x,v_{xx},...) \approx 0,$$

of order two. The approximate determining (3.3) is equivalent to the condition

$$[\mathfrak{B}_0^{(2)}\Xi_0(v) + \epsilon(\mathfrak{B}_1^{(2)}\Xi_0(v) + \mathfrak{B}_0^{(2)}\Xi_1(v))]\big|_{\Xi(v,\epsilon)\approx 0} = O(\epsilon), \tag{3.4}$$

where the exponent $\mathfrak{B}_j^{(2)}$ for $j = 0, 1$ is the second prolongation for the equation, provided by the structure

$$\mathfrak{B}_j^{(2)} = \mathfrak{B}_j + \varphi_j^x\frac{\partial}{\partial v_x} + \varphi_j^t\frac{\partial}{\partial v_t} + \varphi_j^{xx}\frac{\partial}{\partial v_{xx}} + \varphi_j^{xt}\frac{\partial}{\partial v_{xt}} + \varphi_j^{tt}\frac{\partial}{\partial v_{tt}}, \tag{3.5}$$

where $\varphi_j^x, \varphi_j^t, \varphi_j^{xx}, \varphi_j^{xt}, \varphi_j^{tt}$ are detailed [33] in general via relations:

$$\begin{aligned}&\varphi^i = D_i(\varphi) - v_j D_i(\xi^j), \quad \xi^j \text{ is the infinitesimal relative to the independent variables.}\\ &\varphi^{ij} = D_j(\varphi^i) - v_{ik}D_j(\xi^k), \; i,j,k = 1,2,...\end{aligned} \tag{3.6}$$

D_i being the Lie derivative defined as $D_i = \frac{\partial}{\partial x_i} + v_i\frac{\partial}{\partial v} + v_{ij}\frac{\partial}{\partial v_{ij}} + ...$, holding likewise for D_j. Reverting the terms in (3.6) back in (3.5), expanding and distributing the result across different differentials of v, one obtains an algebraic system. Solving this, we have ς_i, φ_i and ϖ_i for $i = 0, 1$.

3.2 Exact Symmetry Analysis

Theorem 2 *Considering the corresponding infinitesimal generator given in (3.2), exact symmetry of the unperturbed term in (2.3) is represented by $\mathfrak{B}_0$, where the infinitesimals are given as follows:*

$$\begin{aligned}\varsigma_0 &= \mathbb{F}_1 - \frac{1}{(\gamma-1)^2}\left(\int_x \frac{\Gamma_1\left(e^{-\frac{(\gamma+1)((a+x)\gamma+x-a)\beta\delta}{\alpha(\gamma-1)}}\right)}{\delta\beta\left(-\gamma e^{\frac{\gamma\beta\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}} + e^{\frac{\delta\beta(\gamma^2x+a\gamma-a+x)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{1}{\gamma-1}}\right)}da\right),\\ \varpi_0 &= \mathbb{F}_2 + \frac{1}{(\gamma-1)^2}\int_x \frac{(\kappa_8 - \kappa_7\gamma - 2\kappa_{11}\gamma + 3\kappa_9(\gamma-1) + 2\kappa_{10})\,\Gamma_2}{\alpha\beta\delta\left(-\gamma\sqrt{\kappa_{10}}) + \sqrt{\kappa_{11}}\right)^2}da,\\ \varphi_0 &= \mathbb{F}_3(x,t,v).\end{aligned} \tag{3.7}$$

where

- $\Gamma_1 = \delta\beta(\gamma-1)(\kappa_1\gamma - \kappa_2\gamma + \kappa_3 - \kappa_4)\mathbb{F}_3 - \alpha(\kappa_1(\gamma+2) + (-2\gamma-1)\kappa_3 + \kappa_2\gamma - \kappa_4)D_1(\mathbb{F}_3) - \delta\beta(\gamma-1)^2(-\kappa_3\gamma + \kappa_1)D_3(\mathbb{F}_3) - 3(\kappa_1 - \kappa_3 + \frac{\kappa_2}{3} - \frac{\kappa_4}{3})D_2(\mathbb{F}_3)$.

- $\Gamma_2 = (\gamma-1)\,\beta\delta\mathbb{F}_3 + (\gamma-1)^2\,\beta\delta\,(\kappa_{11} - (\gamma+1)\,\kappa_9 + \kappa_{10}\gamma)\,D_3\,(\mathbb{F}_3) + \alpha(-\kappa_8 - \kappa_7\gamma + \kappa_{11}(\gamma+3-3\kappa_9(\gamma+1)+3(\gamma+\frac{1}{3})\kappa_{10})D_1(\mathbb{F}_3) + 4D_2(\mathbb{F}_3)(\kappa_{11}+\kappa_{10} - \frac{\kappa_7}{4} - \frac{3\kappa_9}{2} - \frac{\kappa_8}{4})$.
- $\mathbb{F}_1 \equiv \mathbb{F}_1\left(\frac{\alpha\log\left(\frac{v-1}{v-\gamma}\right)}{\beta(\gamma-1)\delta} - x,\ \frac{\ln\left(\frac{v-\gamma}{v-1}\right)+(\gamma-1)\ln\left(\frac{v}{v-1}\right)+\ln(1-\gamma)(\gamma-1)+t\beta\delta\gamma^2-t\beta\delta\gamma}{\delta\beta(\gamma-1)\gamma}\right)$.
- *In* ς_0,

$$\kappa_1 = \left(\frac{v-1}{v-\gamma}\right)^{\frac{1}{\gamma-1}} e^{\frac{2\delta\beta\left(\left(x+\frac{a}{2}\right)\gamma^2+\left(x+\frac{a}{2}\right)\gamma+x-a\right)}{\alpha(\gamma-1)}}; \tag{3.8}$$

$$\kappa_2 = \left(\frac{v-1}{v-\gamma}\right)^{\frac{-1+2\gamma}{\gamma-1}} e^{\frac{3\gamma\beta\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}}; \tag{3.9}$$

$$\kappa_3 = \left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{\left((x+2a)\gamma^2+(4x-a)\gamma+x-a\right)\beta\delta}{\alpha(\gamma-1)}}; \tag{3.10}$$

$$\kappa_4 = \left(\frac{v-1}{v-\gamma}\right)^{\frac{2-\gamma}{\gamma-1}} e^{\frac{3\delta\beta\left(\gamma^2x+a\gamma-a+x\right)}{\alpha(\gamma-1)}} \tag{3.11}$$

- $\mathbb{F}_3 = \mathbb{F}_3(a, \kappa_5, \kappa_6)$ *in* $\Gamma_{1,2}$ *with*

$$\kappa_5 = \frac{1}{\delta\beta\,(\gamma-1)\,\gamma}\left(t\beta\delta\,\gamma^2 - t\beta\delta\gamma + \gamma\ln\left(\frac{v}{v-1}\right) + \ln(1-\gamma)\,\gamma - \ln(j)\,\gamma - \ln\left(\frac{v}{v-1}\right) + \ln\left(\frac{v-\gamma}{v-1}\right) - \ln(1-\gamma) + \ln(j) - \ln\left(\frac{e^{\frac{(\gamma-1)(x-a)\delta\beta}{\alpha}}\,(v-\gamma)}{v-1}\right)\right),$$

where

$$j = \frac{e^{\frac{(\gamma-1)(x-a)\delta\beta}{\alpha}}\,(v-\gamma) - \gamma\,(v-1)}{v-1};$$

$$\kappa_6 = \frac{\left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}}\left((-v+\gamma)\,e^{\frac{\delta\beta\left(\gamma^2x+a\gamma-a+x\right)}{\alpha(\gamma-1)}} + \gamma e^{\frac{\gamma\beta\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}}\,(v-1)\right)}{(v-1)\left(e^{\frac{\gamma\beta\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}} - e^{\frac{\delta\beta\left(\gamma^2x+a\gamma-a+x\right)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{1}{\gamma-1}}\right)}. \tag{3.12}$$

- *In* ϖ_0,

$$\begin{aligned}
\kappa_7 &= \left(\frac{v-1}{v-\gamma}\right)^{\frac{-1+3\gamma}{\gamma-1}} e^{-\frac{\beta\delta\left((x-3a)\gamma^2+(-6x+4a)\gamma+x-a\right)}{\alpha(\gamma-1)}}\\
\kappa_8 &= \left(\frac{v-1}{v-\gamma}\right)^{\frac{3-\gamma}{\gamma-1}} e^{\frac{3\left(\left(x-\frac{a}{3}\right)\gamma^2+\left(-\frac{2x}{3}+\frac{4a}{3}\right)\gamma+x-a\right)\beta\delta}{\alpha(\gamma-1)}};\\
\kappa_9 &= \left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma+1}{\gamma-1}} e^{\frac{(\gamma+1)((a+x)\gamma+x-a)\beta\delta}{\alpha(\gamma-1)}};\\
\kappa_{10} &= \left(\frac{v-1}{v-\gamma}\right)^{\frac{2\gamma}{\gamma-1}} e^{\frac{2\gamma\beta\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}};\\
\kappa_{11} &= \left(\frac{v-1}{v-\gamma}\right)^{\frac{2}{\gamma-1}} e^{\frac{2\delta\beta\left(\gamma^2x+a\gamma-a+x\right)}{\alpha(\gamma-1)}}.
\end{aligned} \tag{3.13}$$

Proof For obtaining exact symmetry $\mathfrak{B}_0$ of the unperturbed equation, we solve the determining equation

$$\mathfrak{B}_0^{(2)}\left(v_t+\alpha v\left(v_x\right)-\delta\beta v^2+\delta\beta v\gamma+\delta\beta v^3-\delta\beta v^2\gamma\right)\Big|_{v_t+\alpha v(v_x)-\delta\beta v^2+\delta\beta v\gamma+\delta\beta v^3-\delta\beta v^2\gamma=0} = 0. \tag{3.14}$$

From (3.14) one achieves a system of equations in derivatives of $\varsigma,\ \varpi,\ \varphi$, from which, with details of Remark 3.1, we achieve (3.7) with values given in (3.8)-(3.13). □

Remark 3.2 Confine $\mathbb{F}_i$, $i=1,2$ to distinct constants c_1, c_4 and consider $\mathbb{F}_3(x,t,v)$ to be a linear function in the first argument, specifically, $\mathbb{F}_3(x,t,v)=c_2x$, c_2 being a constant. By applying further simplifications as in (3.7) followed by simplifying the integral terms within, we obtain:

$$\varsigma_0=\frac{\beta c_1\left(v-1\right)\left(v-\gamma\right)\delta-\alpha x c_2}{\beta\delta\left(v-1\right)\left(v-\gamma\right)},\ \ \varpi_0=c_4-\frac{xc_2}{v\left(v-1\right)\beta\delta\left(v-\gamma\right)},\ \ \varphi_0=c_2x.$$

Definition 3.1 [32] For the unperturbed equation ($\varXi_0$), $\mathfrak{B}_0$ is the stable symmetry in accordance with the terms of Theorem.2. The deformation of $\mathfrak{B}_0$ caused by $\varXi_1$ is represented by corresponding approximate symmetry operator $\mathfrak{B}=\mathfrak{B}_0+\epsilon\mathfrak{B}_1$. In case Lie algebra of symmetry in $\varXi_0=0$ remains stable, perturbed equation ($\varXi$) inherits the symmetries of $\varXi_0$.

3.3 Approximate Symmetry Analysis

Utilizing the results obtained in Subsection 3.2 along with the (3.4) and by splitting the equation obtained with respect to derivatives of the dependent variable, we have the determining equations for $\varsigma_1,\ \varpi_1,$ and φ_1. Solving these, we have:

$$\begin{aligned}
\varsigma_1 &= \mathbb{F}_4-\frac{1}{\left(\gamma-1\right)^2\delta\beta\left(v-\gamma\right)^2\left(v-1\right)}\int_x\frac{1}{\left(-\gamma e^{\frac{\beta\gamma\delta(a\gamma-a+2x)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}}+\omega_1\right)}\\
&\quad\times\omega_1\left(\left(\gamma-1\right)\delta\omega_5\beta\varphi_1+2\omega_4\alpha\mathrm{D}_1(\varphi_1)+\left(\gamma-1\right)^2\delta\left(v-1\right)\beta\left(v-\gamma\right)\right.\\
&\quad\left.\times\left(\gamma\left(v-1\right)\omega_2^{-1}-v+\gamma\right)\mathrm{D}_3(\varphi_1)+3\mathrm{D}_2(\varphi_1)\omega_3\right)da,\\
\varpi_1 &= \mathbb{F}_5+\frac{\omega_1^2}{\left(\gamma-1\right)^2\alpha\beta\delta\left(v-\gamma\right)^3\left(v-1\right)\left(-\gamma\omega_6+\omega_1\right)^2}\int_x\left[-2\left(\gamma-1\right)\delta\right.\\
&\quad\times\left(\omega_2\left(v-\gamma\right)-v+1\right)^2\left(-\frac{1}{2}\left(v-\gamma\right)^2\omega_2^2+\left(\gamma-1\right)\left(v-1\right)\left(v-\gamma\right)\omega_2+\frac{1}{2}\gamma\left(v-1\right)^2\right)\beta\varphi_1\\
&\quad-\omega_7\omega_2^3+\alpha\left(\omega_2\left(v-\gamma\right)-v+1\right)^3\times\left(\left(-v+\gamma\right)\omega_2+\gamma\left(v-1\right)\right)\mathrm{D}_1\varphi_1\\
&\quad\left.-\mathrm{D}_2\varphi_1\left(\omega_2\left(v-\gamma\right)-v+1\right)^4\right]da,\\
\varphi_1 &= \varphi_1(x,t,v),
\end{aligned} \tag{3.15}$$

where

- $\mathbb{F}_4$ and $\mathbb{F}_5$ are functions of same arguments as $\mathbb{F}_1$;

  $\varphi_1\equiv\varphi_1(a,\kappa_5,\kappa_6)$ in ς_1 and ϖ_1

$$\omega_1 = \left(\frac{v-1}{v-\gamma}\right)^{\frac{1}{\gamma-1}} \mathrm{e}^{\frac{\beta\delta\left(\gamma^2 x + a\gamma - a + x\right)}{\alpha(\gamma-1)}};$$
$$\omega_2 = \mathrm{e}^{\frac{(\gamma-1)(x-a)\beta\delta}{\alpha}};$$
$$\omega_3 = \frac{\left(\omega_2\left(v-\gamma\right)-v+1\right)^3}{3\omega_2^2};$$
$$\omega_4 = -\frac{\left(\left(-v+\gamma\right)\omega_2+\gamma\left(v-1\right)\right)\left(\omega_2\left(v-\gamma\right)-v+1\right)^2}{2\omega_2^2};$$
$$\omega_5 = \frac{\left(\omega_2\left(v-\gamma\right)+v-1\right)\left(\left(-v+\gamma\right)\omega_2+\gamma\left(v-1\right)\right)\left(\omega_2\left(v-\gamma\right)-v+1\right)}{\omega_2^2};$$
$$\omega_6 = \mathrm{e}^{\frac{\gamma\delta\beta(a\gamma-a+2x)}{\alpha(\gamma-1)}}\left(\frac{v-1}{v-\gamma}\right)^{\frac{\gamma}{\gamma-1}};$$
$$\omega_7 = \frac{\alpha\left(\omega_2\left(v-\gamma\right)-v+1\right)^3\left(\left(-v+\gamma\right)\omega_2+\gamma\left(v-1\right)\right)\mathrm{D}_1(\varphi_1)}{3\omega_2^3}.$$

By applying analogous manipulations to those used for exact symmetries in (3.15), we derive the following:

$$\varsigma_1 = c_7 + \frac{c_6\alpha x\left(2v^2-2v\gamma+\gamma^2-2v+1\right)}{\left(-v+\gamma\right)\left(v-1\right)\beta\delta\left(\gamma-1\right)^2},$$
$$\varpi_1 = c_5 + \frac{c_6 x\left(\gamma-3\right)}{\delta\beta\left(\gamma-1\right)^2} + \frac{c_6 x}{\delta\beta\left(v-1\right)\left(-v+\gamma\right)v},$$
$$\varphi_1 = c_6 x.$$

Thus, infinitesimal approximate symmetries for (2.3) constitute a five dimensional approximate Lie algebra generated through

$$\varsigma = c_1 + \epsilon\left(c_7 + \frac{c_6\alpha x\left(2v^2-2v\gamma+\gamma^2-2v+1\right)}{\left(-v+\gamma\right)\left(v-1\right)\beta\delta\left(\gamma-1\right)^2}\right),$$
$$\varpi = c_4 + \epsilon\left(c_5 + \frac{c_6 x\left(\gamma-3\right)}{\delta\beta\left(\gamma-1\right)^2} + \frac{c_6 x}{\delta\beta\left(v-1\right)\left(-v+\gamma\right)v}\right),$$
$$\varphi = \epsilon c_6 x.$$

Approximate symmetries of transformed SPGBHE:

$$\mathfrak{B}_1 = \frac{\partial}{\partial x},\ \mathfrak{B}_2 = \frac{\partial}{\partial t},\ \mathfrak{B}_3 = \epsilon\frac{\partial}{\partial x},\ \mathfrak{B}_4 = \epsilon\frac{\partial}{\partial t},$$
$$\mathfrak{B}_5 = \frac{x\epsilon\mathrm{D}\left(\gamma^2-2\gamma v+2v^2-2v+1\right)}{\left(-v+\gamma\right)\left(v-1\right)}\frac{\partial}{\partial x} + \left(\epsilon A x + \frac{\epsilon x}{B\left(v-1\right)\left(-v+\gamma\right)v}\right)\frac{\partial}{\partial t} + x\epsilon\frac{\partial}{\partial v} \tag{3.16}$$

with $A = \frac{\gamma-3}{(\gamma-1)^2\delta\beta}$, $B = \frac{1}{\delta\beta}$, $D = \frac{\alpha}{(\gamma-1)^2\delta\beta}$. Taking the first order of precision into account, next subsection demonstrates that the operators above generate a 5-dimensional approximate symmetry Lie algebra $(\mathfrak{g})$.

3.4 Optimal System

Categorizing the subalgebra in symmetry algebra as distinct conjugate classes via adjoint action associated with group is crucial for systematically performing symmetry reductions of the governing equation. This classification is essential for obtaining group invariant solutions of the concerning PDE. The goal is to obtain optimal set for subalgebras through selecting a single representative element from each category of equivalent subalgebras [34]. For the five dimensional approximate symmetry algebra $\mathfrak{g}$, the commutation relations are

$$\begin{aligned}&[\mathfrak{B}_1, \mathfrak{B}_5] = \mathfrak{B}_5, \quad [\mathfrak{B}_5, \mathfrak{B}_1] = -\mathfrak{B}_5,\\ &\left[\mathfrak{B}_i, \mathfrak{B}_j\right] = 0, \quad \forall\, i, \;\; j \neq 1, \; 5.\end{aligned} \tag{3.17}$$

Here, $\mathfrak{B}_i's$ are as in (3.16)

- Calculating invariant functions: The invariant functions demonstrate the non-equivalence of subalgebras in a 1-dimensional optimal classification. Considering generic elements, $X = \sum_{i=1}^{5} a_i' X_i$ and $Y = \sum_{i=1}^{5} b_i' X_i$, belonging to the span $\{X_i\}$, where a_i' and b_i' may provide coordinates for basis related to $\mathfrak{g}$, adjoint transformation expressed via commutators:

$$\begin{aligned}Ad(exp(\kappa\mathfrak{B}_i))\mathfrak{B}_j &= \mathfrak{B}_j - \kappa[\mathfrak{B}_i, \mathfrak{B}_j] + \frac{\kappa^2}{2}[\mathfrak{B}_i, [\mathfrak{B}_i, \mathfrak{B}_j]] - ...,\\ &= (a_1\mathfrak{B}_1 + a_2\mathfrak{B}_2 + a_3\mathfrak{B}_3 + a_4\mathfrak{B}_4 + a_5\mathfrak{B}_5) -\\ &\quad \kappa(a_1\Upsilon_1 + a_2\Upsilon_2 + a_3\Upsilon_3 + a_4\Upsilon_4 + a_5\Upsilon_5) + O(\kappa^2),\end{aligned}$$

  where $\mathfrak{B}_i, \; \mathfrak{B}_j$ vary from $i, j = 1, ..., 5$ and their Lie bracket is the commutator of $\mathfrak{g}$ from (3.17). Here, $\Psi_5 = a_1 b_5 - a_5 b_1$ and rest all $\Psi_i = 0$. Extract the coefficients of b_i from $\Psi_1 \frac{\partial\phi}{\partial a_1} + \Psi_2 \frac{\partial\phi}{\partial a_2} + \Psi_3 \frac{\partial\phi}{\partial a_3} + \Psi_4 \frac{\partial\phi}{\partial a_4} + \Psi_5 \frac{\partial\phi}{\partial a_5} = 0$, we get a system of PDEs, which, on solving yields the invariant function $\phi = F(a_1, a_2, a_3, a_4)$.
- Adjoint transformation matrix: The adjoint representation for symmetry transformations is necessary to find inequivalent optimal subalgebras. Table 1 provides this representation and facilitates adjoint transformations using $\mathrm{Ad}_{\exp}(\rho\mathfrak{B}_i)\mathfrak{B}_j = e^{-\rho\mathfrak{B}_i}\mathfrak{B}_j e^{\rho\mathfrak{B}_i}$ for the Lie algebra $\mathfrak{g}$, with transformation matrices for each point-symmetry $\mathfrak{B}_i$. The global adjoint transformation matrix results from the product of these matrices and represented as

$$Adj_{\mathfrak{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \rho_5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-\rho_1} \end{bmatrix}$$

Table 1 Adjoint representation table for the singularly perturbed (2.3)

$[\mathfrak{B}_i, \mathfrak{B}_j]$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$\mathfrak{B}_5$
$\mathfrak{B}_1$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$e^{\rho}\mathfrak{B}_5$
$\mathfrak{B}_2$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$\mathfrak{B}_5$
$\mathfrak{B}_3$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$\mathfrak{B}_5$
$\mathfrak{B}_4$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$\mathfrak{B}_5$
$\mathfrak{B}_5$	$\mathfrak{B}_1 - \rho\mathfrak{B}_5$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$	$\mathfrak{B}_5$

- 1-d optimal system: The adjoint transformation equation is given by the relation: $(b_1, b_2, b_3, b_4, b_5) = (a_1, a_2, a_3, a_4, a_5)Adj_{\mathfrak{g}}$. By applying adjoint maps and taking into account all alternatives that must have solutions for ρ_1, ..., ρ_5, the generic element $\mathfrak{B} = \sum_{k=1}^{5} a_k \mathfrak{B}_k$ may be reduced to a simpler form. By using a_1, a_2, a_3, a_4, a_5 as invariants and assigning each one as 1 or 0, we derive the optimal system of the $1-d$ subalgebra as follows:
$$\{\mathfrak{B}_1,\ a_2\mathfrak{B}_2,\ a_3\mathfrak{B}_3 + a_4\mathfrak{B}_4,\ \mathfrak{B}_1 + \mathfrak{B}_4\}.$$

4 Symmetry Reductions and Analytical Solutions

4.1 Reductions Corresponding to (2.3)

Reduction 1: For the generator $\mathfrak{B}_1$, $v(x,t) = \zeta(\mathring{\kappa})$, where $\mathring{\kappa} = x$. Gaining the Lagrange structure through the application of characteristic approach enables us to express this variable. Reverting it back in the (2.3), we get the reduction equation which is the transformed second order ordinary differential equation (ODE)

$$\alpha\zeta^2\zeta' - \delta\beta\zeta^3 + \delta\beta\zeta^2\gamma + \delta\beta\zeta^4 - \delta\beta\zeta^3\gamma + \epsilon\left(-\zeta\zeta'' - \left(\frac{1}{\delta} - 1\right)\zeta'^2\right) = 0, \quad \zeta' = \frac{d\zeta}{dx}. \tag{4.1}$$

Reduction 2: Corresponding to the generator $b\mathfrak{B}_2$, (b denotes a_2) we have similarity variable as $\mathring{\kappa} = bt$ and similarity solution is $v(x,t) = \frac{1}{b}\zeta(\mathring{\kappa})$. After substituting into (2.3), the resulting simplified form of reduction is defined as:

$$b^3\zeta' - b\delta\beta\zeta^2 + b^2\delta\beta\zeta\gamma + \delta\beta\zeta^3 - b\delta\beta\zeta^2\gamma = 0. \tag{4.2}$$

Reduction 3. For the generator $a\mathfrak{B}_3 + b\mathfrak{B}_4$, Lagrange form of PDE is achieved after getting the characteristic equation. Similarity variables associated with generator are $v(x,t) = \zeta(\mathring{\kappa})$, $\mathring{\kappa} = ax - bt$, resulting in the transformation $v(x,t) = \zeta(ax - bt)$ which when applied on the governing SPPDE yields a more manageable form of (2.3). The simplified form obtained is thus a second order nonlinear singularly perturbed ODE mentioned below.

$$-b\zeta\zeta' + \alpha d\zeta^2\zeta' - \delta\beta\zeta^3 + \delta\beta\zeta^2\gamma + \delta\beta\zeta^4 - \delta\beta\zeta^3\gamma + \epsilon(a)^2\left(\zeta\zeta'' - \left(\frac{1}{\delta} - 1\right)\zeta'^2\right) = 0, \tag{4.3}$$

where $\zeta' = \dfrac{d\zeta}{d\mathring{\kappa}}$.

4.2 Analytical Solutions Corresponding to (2.1)

Next, we focus on finding solutions of transformed ODEs and hence of (2.3). Substituting the transformation $u = v^{\frac{1}{\delta}}$ back into the equation enables one to obtain the solutions of the governing (2.1).

An Overview of the tanh-coth Procedure

Herein, we briefly offer a simplified illustration of the tanh-coth method [35].

(i) Consider a nonlinear PDE in its general form $P(v, v_t, v_x, v_{xx}, ...,) = 0$. To change the PDE into ODE, we introduce the smilarity variable κ (wave form conversion) obtaining $v(x,t) = \zeta(\kappa)$ to get $O(\zeta, \zeta^{'}, \zeta'', ...) = 0$.

(ii) For the purpose of representing the derivatives, there is an introduction of new independent variable, $T = tanh(\mu\kappa)$,

$$\begin{aligned} \frac{d}{d\kappa} &= \mu(1-T^2)\frac{d}{dT}, \\ \frac{d^2}{d\kappa^2} &= -2\mu^2 T(1-T^2)\frac{d}{dT} + \mu^2(1-T^2)^2\frac{d^2}{dT^2}, \end{aligned} \tag{4.4}$$

similarly finding out the other derivatives.

(iii) Assume the solution to the ODE is expressed as a finite series

$$\zeta(\mu\kappa) = U(T) = \sum_{\phi=0}^{m} a_\phi T^\phi + \sum_{\phi=1}^{m} b_{-\phi} T^{-\phi}, \tag{4.5}$$

where the determination of the specific positive integer m is done by the homogeneous balancing principle.

(iv) Reverting back the values obtained by (4.4) and (4.5) in the ODE followed by assembling the coefficients of like powers of T leads to an algebraic system. This system can be solved for the desired constants. Substituting the values back, we acquire solution to NLPDE.

4.2.1 Exact Analytical Solutions Corresponding to (4.1)

By balancing the terms as mentioned above, $m = 1$. So, (4.5) is $b_0 + b_1 T(\mathring{k})$. System of equations is :

$$\begin{aligned} T^0 &: \alpha {b_0}^2 b_1 k - \delta\beta {b_0}^3 + \delta\beta {b_0}^2\gamma + \delta\beta {b_0}^4 - \delta\beta {b_0}^3\gamma - \epsilon\left(\frac{1}{\delta}-1\right){b_1}^2k^2 = 0, \\ &\vdots \\ T^4 &: \alpha {b_1}^3 k + \delta\beta {b_1}^4 + \epsilon\left(-2{b_1}^2k^2 - \left(\frac{1}{\delta}-1\right){b_1}^2k^2\right) = 0. \end{aligned} \tag{4.6}$$

Solving the system in (4.6), we get

$$\begin{aligned} u_1(x,t,\epsilon) &= 1, \\ u_2(x,t,\epsilon) &= 0, \\ u_3(x,t,\epsilon) &= \gamma. \end{aligned} \tag{4.7}$$

4.2.2 Exact Analytical Solutions Corresponding to (4.2)

The ODE in reduction (2) has the implicit solution of (2.1) provided below:

$$\frac{b\ln\left(b\left(\gamma-(u(x,t))^\delta\right)\right) - b\ln\left(b\left((u(x,t))^\delta-1\right)\right)\gamma + \left(b\ln\left(bu(x,t)^\delta\right) + \beta\delta\gamma\,(bt+c_1)\right)(\gamma-1)}{\beta\delta\gamma\,(\gamma-1)}, \tag{4.8}$$

where c_1 denotes a constant.

4.2.3 Exact Analytical Solutions Corresponding to (4.3)

Much like determining the solutions for (4.1), our task now is to identify the solutions for reduction 3. Apply the homogeneous balance principle to (4.3). Substitute the trial solution $\zeta = a_0 + a_1 T(\theta) + b_1 T(\theta)^{-1}$ and T' into (4.3). By gathering the terms of similar powers of T, we obtain:

$$
\begin{aligned}
T^{-4} &: \frac{b_1^2\left(-\delta^2\beta b_1^2 + \alpha a b_1 k\delta + \epsilon a^2 k^2 (\delta+1)\right)}{\delta} = 0,\\
T^{-3} &: 2b_1\left(\frac{\beta\delta(\gamma - 4a_0 + 1)\, b_1^2}{2} + k\left(a\alpha a_0 - \frac{b}{2}\right) b_1 + a^2\epsilon k^2 a_0\right) = 0,\\
&\vdots\\
T^{4} &: \frac{\left(-\delta^2\beta b_0^2 + \alpha a b_0 k\delta + \epsilon a^2 k^2 (\delta+1)\right) b_0^2}{\delta} = 0.
\end{aligned}
$$

Through the process of solving this problem, we are able to obtain distinct sets of solution for the variables a_0, a_1, b_1, a, b and k. These answers are subsequently substituted into the trial solution. Therefore, we obtain:

$$
u_{1,2}(x,t,\epsilon) = \left(\frac{1}{2} \mp \frac{\tanh(k(ax - bt))}{2}\right)^{\frac{1}{\delta}}, \tag{4.9}
$$

where for u_1 we have:

$$
\left\{
\begin{array}{l}
a = \pm\frac{\alpha\delta - \delta\sqrt{4\beta\delta\epsilon + \alpha^2 + 4\beta\epsilon}}{4\delta\epsilon k + 4\epsilon k},\\
b = \pm\frac{\delta\left(\alpha^2 - 2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon - \alpha\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}.
\end{array}
\right\}
$$

For u_2:

$$
\left\{
\begin{array}{l}
a = \pm\frac{\delta\left(\alpha + \sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon},\\
b = \pm\frac{\delta\left(\alpha^2 - 2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon + \alpha\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}.
\end{array}
\right\}
$$

Substituting the values of a and b derived from the aforementioned sets into equation (4.9) results in solutions of (2.1).

Next,

$$
u_{3,4}(x,t,\epsilon) = \left(\frac{\gamma}{2} \mp \frac{\gamma\tanh(k(ax - bt))}{2}\right)^{\frac{1}{\delta}}, \tag{4.10}
$$

for u_3, we have the values of a and b as below:

$$
\left\{
\begin{array}{l}
a = \pm\frac{\gamma\delta\left(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon},\\
b = \pm\frac{\gamma\delta\left(\alpha^2\gamma + 2\beta(-1+\gamma-\delta)(1+\delta)\epsilon - \alpha\gamma\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}.
\end{array}
\right\}
$$

and u_4:

$$
\left\{
\begin{array}{l}
a = \pm\frac{\gamma\delta\left(\alpha + \sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon},\\
b = \pm\frac{\gamma\delta\left(\alpha^2\gamma + 2\beta(-1+\gamma-\delta)(1+\delta)\epsilon + \alpha\gamma\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}.
\end{array}
\right\}
$$

Hence, by substituting the values from the aforementioned sets into (4.10), we obtain solutions for (2.1). Proceeding further,

$$u_{5,6}(x,t,\epsilon)=\left(\frac{1}{2}\mp\frac{\coth(k(ax-bt))}{2}\right)^{\frac{1}{\delta}}, \tag{4.11}$$

The values of the unknowns to be substituted into (4.11) in order to obtain solutions corresponding to (2.1) are as follows:

For u_5:

$$\left\{\begin{array}{l} a=\pm\frac{\alpha\delta-\delta\sqrt{4\beta\delta\epsilon+\alpha^2+4\beta\epsilon}}{4\delta\epsilon k+4\epsilon k}, \\ b=\pm\frac{\delta\left(\alpha^2-2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon-\alpha\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}. \end{array}\right\}$$

and for u_6:

$$\left\{\begin{array}{l} a=\pm\frac{\delta\left(\alpha+\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon}, \\ b=\pm\frac{\delta\left(\alpha^2-2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon+\alpha\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}. \end{array}\right\}$$

Continuing with the analysis, we have:

$$u_{7,8}(x,t,\epsilon)=\left(\frac{\gamma}{2}\mp\frac{\gamma\coth(k(ax-bt))}{2}\right)^{\frac{1}{\delta}}, \tag{4.12}$$

where the set of values for u_7 are

$$\left\{\begin{array}{l} a=\pm\frac{\gamma\delta\left(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon}, \\ b=\pm\frac{\gamma\delta\left(\alpha^2\gamma+2\beta(-1+\gamma-\delta)(1+\delta)\epsilon-\alpha\gamma\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}. \end{array}\right\}$$

The set of values for u_8 corresponds to:

$$\left\{\begin{array}{l} a=\pm\frac{\gamma\delta\left(\alpha+\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)\epsilon}, \\ b=\pm\frac{\gamma\delta\left(\alpha^2\gamma+2\beta(-1+\gamma-\delta)(1+\delta)\epsilon+\alpha\gamma\sqrt{\alpha^2+4\beta(1+\delta)\epsilon}\right)}{4k(1+\delta)^2\epsilon}. \end{array}\right\}$$

These sets when substituted back into (4.12) represents the solution of (2.1). Next, u_9 is given by

$$u_9(x,t,\epsilon)=\left(\frac{1}{2}\pm\frac{\tanh(k(ax-bt))}{4}\pm\frac{\coth(k(ax-bt))}{4}\right)^{\frac{1}{\delta}}, \tag{4.13}$$

where

$$\left\{\begin{array}{l} a=\mp\frac{k\alpha\delta+\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}}{8k^2(1+\delta)\epsilon}, \\ b=\mp\frac{\alpha\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}+k\delta\left(\alpha^2-2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon\right)}{8k^2(1+\delta)^2\epsilon}. \end{array}\right\}$$

The solution

$$u_{10}(x,t,\epsilon)=\left(\frac{1}{2}\mp\frac{\tanh(k(ax-bt))}{4}\mp\frac{\coth(k(ax-bt))}{4}\right)^{\frac{1}{\delta}}, \tag{4.14}$$

with a, b as below:

$$\left\{\begin{array}{l} a=\pm\frac{k\alpha\delta-\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}}{8k^2(1+\delta)\epsilon}, \\ b=\pm\frac{-\alpha\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}+k\delta\left(\alpha^2-2\beta(1+\delta)(\delta\gamma+\gamma-1)\epsilon\right)}{8k^2(1+\delta)^2\epsilon}. \end{array}\right\}$$

Solution u_{11} of this reduction is provided by:

$$u_{11}(x,t,\epsilon)=\left(\frac{\gamma}{2}\pm\frac{\gamma\tanh(k(ax-bt))}{4}\pm\frac{\gamma\coth(k(ax-bt))}{4}\right)^{\frac{1}{\delta}}, \tag{4.15}$$

having

$$\left\{\begin{array}{l} a=\mp\frac{k\alpha\gamma\delta+\gamma\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}}{8k^2(1+\delta)\epsilon}, \\ b=\mp\frac{\gamma\left(\alpha\gamma\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}+k\delta\left(\alpha^2\gamma+2\beta(-1+\gamma-\delta)(1+\delta)\epsilon\right)\right)}{8k^2(1+\delta)^2\epsilon}. \end{array}\right\}$$

Next solution,

$$u_{12}(x,t,\epsilon)=\left(\frac{\gamma}{2}\mp\frac{\gamma\tanh(k(ax-bt))}{4}\mp\frac{\gamma\coth(k(ax-bt))}{4}\right)^{\frac{1}{\delta}}, \tag{4.16}$$

for

$$\left\{\begin{array}{l} a=\pm\frac{k\alpha\gamma\delta-\gamma\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}}{8k^2(1+\delta)\epsilon}, \\ b=\pm\frac{\gamma\left(-\alpha\gamma\sqrt{k^2\delta^2\left(\alpha^2+4\beta(1+\delta)\epsilon\right)}+k\delta\left(\alpha^2\gamma+2\beta(-1+\gamma-\delta)(1+\delta)\epsilon\right)\right)}{8k^2(1+\delta)^2\epsilon}. \end{array}\right\}$$

Equations (4.7), (4.8), (4.9)-(4.16) are exact solutions.

4.2.4 Approximate Solutions

The approximate invariant

$$I(x,t,v,\epsilon)=I_0(x,t,v,\epsilon)+\epsilon I_1(x,t,v,\epsilon)+o(\epsilon)$$

of the generator $\tilde{\mathfrak{B}}=\mathfrak{B}_1+\mathfrak{B}_4=\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial t}=\tilde{\mathfrak{B}}^0+\epsilon\tilde{\mathfrak{B}}^1$ is provided by the equation

$$\tilde{\mathfrak{B}}(I)=o(\epsilon) \tag{4.17}$$

Equation (4.17) is equivalent to

$$\tilde{\mathfrak{B}}^0(I^0)=0,$$
$$\tilde{\mathfrak{B}}^0(I^1)+\tilde{\mathfrak{B}}^1(I^0)=0.$$

Solving the aforementioned equation, the independent approximate invariant solutions are $\theta=\frac{at^2}{2}-\epsilon xt$ and $\mu=v$. Applying the transformation $v(x,t)=\zeta(\frac{at^2}{2}-\epsilon xt)$ to the (2.3) results in a simplified singularly perturbed nonlinear ODE,

$$\begin{aligned}&(at-\epsilon x)\,\zeta\zeta'+\alpha\zeta^2\zeta'(-\epsilon t)-\delta\beta\zeta^3+\delta\beta\zeta^2\gamma+\delta\beta\zeta^4-\delta\beta\zeta^3\gamma+\\&\epsilon\left(-\zeta\zeta''(-\epsilon t)^2-\left(\frac{1}{\delta}-1\right)\zeta'^2(-\epsilon t)^2\right)=0,\end{aligned} \tag{4.18}$$

where $\zeta=\zeta(\theta)$ and $\zeta'=\frac{d}{d\theta}$. Utilizing the methodology mentioned in Subsection 4.2, henceforth, following homogeneous balance principle, the solution is assumed to be $\zeta(\theta)=$

$a_0 + a_1 tanh(k\theta) + b_1 tanh^{-1}(k\theta)$. The unknowns $a_0,\ a_1,\ k,\ a,\ b_1$ are found by reverting back the assumed solution form in (4.18) and thereby collecting the like powers of $tanh(k\theta)$. This process yields a system of algebraic equations. One gets numerous solution sets after solving the resultant system. Substituting these back in $\zeta(\theta)$ yields the solutions of (2.1) as given below.

$$u_{\varrho_1}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{2}\tanh(\Upsilon_1)\right]^{1/\delta};\ \ u_{\varrho_2}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{2}\coth(\Upsilon_1)\right]^{1/\delta}, \\ \text{where } \Upsilon_1 = \left\{ \frac{(\mathcal{X}+\alpha\delta\varepsilon t)\left(\delta\varepsilon(\alpha t(\gamma\delta+\gamma+1)-2(\delta+1)x)-(\gamma\delta+\gamma-1)\mathcal{X}\right)}{16t\delta(\delta+1)^2\epsilon^3} \right\}. \tag{4.19}$$

$$u_{\varrho_3}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{2}\tanh(\Upsilon_2)\right]^{1/\delta};\ \ u_{\varrho_4}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{2}\coth(\Upsilon_2)\right]^{1/\delta}, \\ \text{where } \Upsilon_2 = \left\{ \frac{(\alpha\delta\varepsilon t-\mathcal{X})\left((\gamma\delta+\gamma-1)\mathcal{X}+\delta\varepsilon(\alpha t(\gamma\delta+\gamma+1)-2(\delta+1)x)\right)}{16t\delta(\delta+1)^2\epsilon^3} \right\}. \tag{4.20}$$

$$u_{\varrho_5}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\Upsilon_3)\right]^{1/\delta};\ \ u_{\varrho_6}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\coth(\Upsilon_3)\right]^{1/\delta} \\ \text{where } \Upsilon_3 = \left\{ \frac{\gamma(\alpha\delta\varepsilon t-\mathcal{X})\left((-\gamma+\delta+1)\mathcal{X}+\delta\varepsilon(\alpha t(\gamma+\delta+1)-2(\delta+1)x)\right)}{16t\delta(\delta+1)^2\epsilon^3} \right\} \tag{4.21}$$

$$u_{\varrho_7}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\Upsilon_4)\right]^{1/\delta};\ \ u_{\varrho_8}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\coth(\Upsilon_4)\right]^{1/\delta}, \\ \text{where } \Upsilon_4 = \left\{ \frac{\gamma\left((\gamma-\delta-1)\mathcal{X}+\delta\varepsilon(\alpha t(\gamma+\delta+1)-2(\delta+1)x)\right)(\mathcal{X}+\alpha\delta\varepsilon t)}{16t\delta(\delta+1)^2\epsilon^3} \right\} \tag{4.22}$$

$$u_{\varrho_9}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{4}\tanh\left(\frac{\Upsilon_2}{2}\right) + \frac{1}{4}\coth\left(\frac{\Upsilon_2}{2}\right)\right]^{1/\delta}; \tag{4.23}$$

$$u_{\varrho_{10}}(x,t,\epsilon) = \left[\frac{1}{2} + \frac{1}{4}\tanh\left(\frac{\Upsilon_1}{2}\right) + \frac{1}{4}\coth\left(\frac{\Upsilon_1}{2}\right)\right]^{1/\delta}; \tag{4.24}$$

$$u_{\varrho_{11}}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{4}\tanh\left(\frac{\Upsilon_3}{2}\right) + \frac{\gamma}{4}\coth\left(\frac{\Upsilon_3}{2}\right)\right]^{1/\delta}; \tag{4.25}$$

$$u_{\varrho_{12}}(x,t,\epsilon) = \left[\frac{\gamma}{2} + \frac{\gamma}{4}\tanh\left(\frac{\Upsilon_4}{2}\right) + \frac{\gamma}{4}\coth\left(\frac{\Upsilon_4}{2}\right)\right]^{1/\delta}; \tag{4.26}$$

In above, $\mathcal{X} = \delta\varepsilon t\sqrt{\left(\alpha^2 + 4\beta(\delta+1)\varepsilon\right)}$. Here, (4.19)-(4.26) represent the approximate solutions of SPGBHE denoted by (2.1).

5 Exponentially Fitted Finite Element Scheme for SPGBHE

Consider (2.1) in the operator form defined as

$$\mathsf{L}_\epsilon u(x,t) \equiv -\epsilon u_{xx} + \alpha u^\delta u_x - \beta u(1-u^\delta)(u^\delta - \gamma) + u_t = 0,\ \forall\,(x,t) \in \Omega \tag{5.1}$$

with $\Omega = (0,1) \times (0,1)$, alongside the conditions

$$\begin{aligned} u(x,0) &= u_0(x), \qquad x \in \bar{\Omega}_x. \\ u(0,t) &= k_0(t), \qquad t \in \bar{\Omega}_t. \\ u(1,t) &= k_1(t), \qquad t \in \bar{\Omega}_t. \end{aligned} \tag{5.2}$$

Equation (5.2) define initial and boundary conditions. The assumption is that these conditions are sufficiently smooth and that the boundary conditions match with initial conditions at the end points of $\bar{\Omega}$, thereby guaranteeing a unique solution. Compatibility conditions are

$$u_0(0) = k_0(0) \text{ and } u_0(1) = k_1(0). \tag{5.3}$$

This section explores the finite element method to address the governing singularly perturbed equation. Given that the problem under consideration given by (2.1) qualifies as singularly perturbed, sharp boundary layers emerge close to $x = 1$ when $\epsilon \to 0$. Aiming to tackle this, temporal semi-discretization is done via explicit scheme and domain discretization is done through piecewise uniform Shishkin mesh.

5.1 Time Semidiscretization

Time discretization is achieved by utilizing the explicit Euler method having uniform time increment, denoted by $\Delta t (= k)$. We divide the temporal domain [0,1] into M subintervals providing $\Delta t = \frac{1}{M}$. Then the discrete time mesh $\bar{\Omega}_{t_m}$ for the time domain $\bar{\Omega}_t$ will be given by $\bar{\Omega}_{t_m} = \{t_m | t_m = m\Delta t,\ m = 0, 1, 2, ..., M\}$.

Now the explicit Euler scheme leads one to:

$$\begin{aligned}
&\frac{u^{m+1} - u^m}{\Delta t} = \epsilon u_{xx}^{m+1} - \alpha (u^\delta)^{m+1} u_x^{m+1} + \beta u^{m+1}(1 - (u^\delta)^{m+1})((u^\delta)^{m+1} - \gamma), \\
&u(x, 0) = u_0(x),\ 0 < x < 1, \\
&u(0, t_{m+1}) = u^{m+1}(0) = \tilde{B}_0(t_{m+1}), \\
&u(1, t_{m+1}) = u^{m+1}(1) = \tilde{B}_1(t_{m+1}),\ m > 0,
\end{aligned} \tag{5.4}$$

where solution of the differential equation during time level $(m + 1)$ is identified as u^{m+1}.

5.2 Quasilinearization Technique

Bellman and Kalaba [36] proposed a powerful method in order to achieve linear approximation of nonlinear differential equations, known as quasilinearization. Given that the governing problem is nonlinear in nature, containing ($u^\delta u_x$ and $u(1 - u^\delta)(u^\delta - \gamma)$), quasilinearization is used to attain linear approximation. Nonlinear problem under consideration is transformed into a series of linearized equations about a nominal solution [37] adhering to specified boundary conditions. Assume the nominal solution as $u^{(s)}(x)$. Application of quasilinearization [38] method to nonlinear (5.4) yields a sequence $\langle u^{(s)} \rangle_{s=0}^{\infty}$ linear equations defined via a recurrence relation followed by the equation actually leading to

$$\begin{aligned}
-\epsilon \Delta t \{u_{s+1}^{m+1}\}_{xx} &+ \alpha \Delta t \{u_s^{m+1}\}\{u_{s+1}^{m+1}\}_x + \{u_{s+1}^{m+1}\}\{1 + \alpha \Delta t \{u_s^{m+1}\}_x \\
&- \beta \Delta t \left(2\{u_s^{m+1}\} - 3\{u_s^{m+1}\}^2 - \gamma + 2\gamma \{u_s^{m+1}\}\right)\} \\
= u_m^{s+1} &+ \alpha \Delta t \{u_s^{m+1}\}\{u_s^{m+1}\}_x + 2\beta \Delta t \{u_s^{m+1}\}^3 - \beta \Delta t (1 + \gamma)\{u_s^{m+1}\}^2
\end{aligned} \tag{5.5a}$$

with boundary conditions as

$$u_{s+1}^{m+1}(0) = \tilde{B}_0((m + 1)\Delta t),\ \ u_{s+1}^{m+1}(1) = \tilde{B}_1((m + 1)\Delta t),\ \ m > 0, \tag{5.5b}$$

where s denotes the iteration index. $u_0^{m+1}(x) = u_0(x)$ represents the initial approximation. Equation (5.5a) has the operator form:

$$\begin{aligned}
&\mathsf{L}_\epsilon u^{m+1} = l(x)\\
\text{where }\ &\mathsf{L}_\epsilon = -\epsilon\partial_{xx} + \tilde{b}(x)\partial_x + \tilde{c}(x),\\
&\tilde{b}(x) = b_s(x, t^{m+1}) = \alpha\{u_s^{m+1}\},\\
&\tilde{c}(x) = c_s(x, t^{m+1}) = \{1 + \alpha\Delta t\{u_s^{m+1}\}_x\\
&\qquad\qquad - \beta\Delta t\left(2\{u_s^{m+1}\} - 3\{u_s^{m+1}\}^2 - \gamma + 2\gamma\{u_s^{m+1}\}\right)\}\\
&l(x) = l_s(x, t^{m+1}) = u_m^{s+1} + \alpha\Delta t\{u_s^{m+1}\}\{u_s^{m+1}\}_x + 2\beta\Delta t\{u_s^{m+1}\}^3\\
&\qquad\qquad - \beta\Delta t(1+\gamma)\{u_s^{m+1}\}^2
\end{aligned} \tag{5.6}$$

Functions $\tilde{b}(x),\ \tilde{c}(x),\ l(x)$ are assumed to have sufficient smoothness along the spatial dimension having

$$0 < \alpha^{**} \le \tilde{b}(x) \le \alpha^*, \quad x \in \bar{\Omega}_x$$

$$0 < \beta^{**} \le \tilde{c}(x) \le \beta^*, \quad x \in \bar{\Omega}_x$$

Next, we require that as $\lim\limits_{s\to\infty} u_s^{m+1}(x) = u^{m+1}(x)$,

$$\left|u_{s+1}^{m+1}(x) - u_{s+1}^{m}(x)\right| < \check{\zeta}, \quad x \in \bar{\Omega}_x, \tag{5.8}$$

where $\check{\zeta}$ is the prescribed tolerance. When (5.8) is satisfied, the desired accuracy is achieved and the iteration process is terminated.

Lemma 3 *(Convergence of the Quasilinearization process). Consider*

$$\epsilon\{u^{m+1}\}_{xx} = F(u^{m+1}), \quad x \in \Omega_x,$$

$$u^{m+1}(0) = \tilde{B}_0(t_{m+1}), \quad u^{m+1}(1) = \tilde{B}_1(t_{m+1}).$$

As a result of quasilinearization, a sequence $\langle u_s^{m+1}\rangle$ comprising linear differential equations is defined through recurrence relation given below.

$$\epsilon\{u_{s+1}^{m+1}\}_{xx} = \bar{F}(u_s^{m+1}) + (u_{s+1}^{m+1} - u_s^{m+1})\frac{\partial\bar{F}(u_s^{m+1})}{u_s^{m+1}}, \quad x \in \Omega_x, \tag{5.10}$$

with boundary conditions given by (5.9). Assume $u_0^{m+1}(x)$ to be an initial approximation. Rewriting (5.10) at the previous iteration step and subtracting the resultant equation from (5.10) results in

$$\begin{aligned}
\epsilon((u_{s+1}^{m+1})_{xx} - (u_s^{m+1})_{xx}) = \bar{F}(u_s^{m+1}) - \bar{F}(u_{s-1}^{m+1}) - (u_s^{m+1} - u_{s-1}^{m+1})\frac{\partial\bar{F}(u_{s-1}^{m+1})}{u_{s-1}^{m+1}}\\
+(u_{s+1}^{m+1} - u_s^{m+1})\frac{\partial\bar{F}(u_s^{m+1})}{u_s^{m+1}}, \quad x \in \Omega_x.
\end{aligned}$$

This represents differential equation in $((u_{s+1}^{m+1})-(u_s^{m+1}))$. *By applying Green's function, it is converted to the following integral equation.*

$$\begin{aligned}\epsilon((u_{s+1}^{m+1})_{xx}-(u_s^{m+1})_{xx}) = \int_0^1 \mathrm{G}(x,\mathbb{S})[\bar{F}(u_s^{m+1})-\bar{F}(u_{s-1}^{m+1}) \\ -(u_s^{m+1}-u_{s-1}^{m+1})\frac{\partial\bar{F}(u_{s-1}^{m+1})}{u_{s-1}^{m+1}} \\ +(u_{s+1}^{m+1}-u_s^{m+1})\frac{\partial\bar{F}(u_s^{m+1})}{u_s^{m+1}}]d\mathbb{S},\ x\in\Omega_x,\end{aligned} \tag{5.11}$$

where

$$\mathrm{G}(x,\mathbb{S}) = \begin{cases} x(\mathbb{S}-1), & 0\le x\le \mathbb{S}\le 1,\\ \mathbb{S}(x-1), & 0\le \mathbb{S}\le x\le 1.\end{cases}$$

Here, $\max_{x,\mathbb{S}} \mathrm{G}(x,\mathbb{S}) = 0.25$. *Applying mean-value theorem,*

$$\begin{aligned}\bar{F}(u_s^{m+1})-\bar{F}(u_{s-1}^{m+1}) = (u_s^{m+1}-u_{s-1}^{m+1})\frac{\partial\bar{F}(u_{s-1}^{m+1})}{u_{s-1}^{m+1}} \\ +\frac{(u_s^{m+1}-u_{s-1}^{m+1})^2}{2}\frac{\partial^2\bar{F}(\theta)}{\partial(u^{m+1})^2},\end{aligned} \tag{5.12}$$

where $u_{s-1}^{m+1}\le\theta\le u_s^{m+1}$.*Using* (5.12), (5.11) *becomes* $\epsilon(u_{s+1}^{m+1}-u_s^{m+1})$

$$= \int_0^1 \mathrm{G}(x,\mathbb{S})\left[\frac{(u_s^{m+1}-u_{s-1}^{m+1})^2}{2}\frac{\partial^2\bar{F}(\theta)}{\partial(u^{m+1})^2}+(u_{s+1}^{m+1}-u_s^{m+1})\frac{\partial\bar{F}(u_s^{m+1})}{u_s^{m+1}}\right]d\mathbb{S}. \tag{5.13}$$

Taking

$$\max_{\|u_s^{m+1}\|\le 1}\frac{\partial\bar{F}(u_s^{m+1})}{u_s^{m+1}} = p, \quad \max_{\|u^{m+1}\|\le 1}\frac{\partial^2\bar{F}(u^{m+1})}{\partial(u^{m+1})^2} = q. \tag{5.14}$$

Taking maximum norm over Ω_x, *using* (5.14) *in* (5.13) *results in an inequality, which after rearrangement of terms yields*

$$\|(u_{s+1}^{m+1}-u_s^{m+1})\|_\Omega \le \frac{q}{8\epsilon-2p}\|(u_s^{m+1}-u_{s-1}^{m+1})\|_\Omega^2.$$

This proves that with appropriate choice of initial iterative approximation, the sequence $\langle u_s^{m+1}\rangle$ *converges quadratically, provided* $\frac{q}{8\epsilon-2p}<1$.

Lemma 4 *(Maximum Principle). Consider some mesh function* $\mathsf{m}(x,t)\in C^{2,1}(\bar{\Omega})$ *such that*

(a)$\mathsf{m}(x,t)$ *is non-negative for all* (x,t) *on the boundary of* $\bar{\Omega}$.

(b)$\mathsf{L}_\epsilon\mathsf{m}(x,t)$ *is non-negative for all* (x,t) *throughout the domain* Ω.

with L_ϵ *defined as in* (5.1), *it follows that* $\mathsf{m}(x,t)\ge 0$ *for all* $(x,t)\in\bar{\Omega}$.

Proof Assume the existence of the points $(\tilde{\mathsf{x}},\tilde{\mathsf{t}})\in\bar{\Omega}$ provided

$$\mathsf{m}(\tilde{\mathsf{x}},\tilde{\mathsf{t}}) = \min_{(x,t)\in\bar{\Omega}}\mathsf{m}(x,t) < 0.$$

Under the presumption of this Lemma, $(\tilde{x}, \tilde{t}) \in \Omega$. Without losing generality, it can be assumed that $\tilde{c}(x) > 0$ and large on $\bar{\Omega}$ (refer [39]). Next, because $(\tilde{x}, \tilde{t})$ represents the minimum point of m,

$$m_{xx}(\tilde{x}, \tilde{t}) > 0, \quad m_x(\tilde{x}, \tilde{t}) = 0, \quad m_t(\tilde{x}, \tilde{t}) = 0.$$

Applying differential operator on m,

$$\implies L_\epsilon m(\tilde{x}, \tilde{t}) = -\epsilon m_{xx}(\tilde{x}, \tilde{t}) + \tilde{b}(\tilde{x}) m_x(\tilde{x}, \tilde{t}) - \tilde{c}(\tilde{x}) m(\tilde{x}, \tilde{t}) + m_t(\tilde{x}, \tilde{t}) < 0.$$

This contradicts the assumption, thereby proving that L_ϵ adheres to maximum principle. □

Lemma 5 *(Continuous stability estimate). Consider $u(x, t)$ as the solution to (2.1), then $\forall$ $\epsilon > 0$, one attains*

$$\|u\|_{\bar{\Omega}} \le t\|u_0\|_{\partial\Omega_x} + \|u\|_{\partial\bar{\Omega}}$$

where $\|.\|$ is the L_∞ norm.

Proof Define barrier functions

$$\Psi^{\pm}(x, t) = t\|u_0\|_{\partial\Omega_x} + \|u\|_{\partial\bar{\Omega}} \pm u(x, t), \ \forall (x, t) \in \partial\Omega$$

Implication from the definition of $\Psi^{\pm}$ concludes that $\Psi^{\pm}(x, t) \ge 0$ for all points in $\partial\Omega$. Furthermore, $L_\epsilon \Psi^{\pm}(x, t) \ge 0$ for all points belonging to $\partial\Omega$. Hence, Lemma 4 implies that $L_\epsilon \Psi^{\pm}(x, t) \ge 0$ for each $(x, t) \in \Omega$. Thus, desired estimate is achieved. □

Lemma 6 [39] *$u(x, t) = u_1(x, t) + \epsilon u_2(x, t) + P(x, t)$, such that u_1 as well as u_2 satisfy parabolic equations analogous to (5.5a) subject to zero initial-boundary conditions. Also, P does not depend on ϵ.*

Lemma 7 *Following the compatibility conditions specified in (5.3), the solution of (5.1) in (5.6) along with (5.2) satisfies $|u(x, t) - u_0(x)| \le Ct$ as well as $|u(x, t) - k_0(t)| \le Cx$, $(x, t) \in \bar{\Omega}$ for certain C not depending on the perturbation parameter ϵ. Furthermore,*

$$|u(x, t)| \le C, \quad \textit{for all } (x, t) \in \bar{\Omega} \tag{5.15}$$

Proof Define $v(x, t)$ as $u(x, t) - u_0(x)$. Then, we have

$$L_\epsilon v(x, t) = -L_\epsilon u_0(x).$$

Smoothness property of u_0 implies $|L_\epsilon v| \le C$. In addition, the compatibility condition implies $v(x, t)$ equals zero, for (x, t) in $\partial\bar{\Omega}$. Choosing a barrier function Ct, $L_\epsilon(Ct) = C \ge L_\epsilon v(x, t)$, $(x, t) \in \Omega$. On $\partial\bar{\Omega}$, Ct is non-negative and equals $v(x, t)$. Henceforth, applying the maximum principle,

$$|u(x, t) - u_0(x)| \le Ct, \quad (x, t) \in \bar{\Omega} \tag{5.16}$$

Going by the similar argument proves another inequality. Equation (5.15) follows from (5.16). □

Lemma 8 *The absolute value of $u_t(x, t)$ is bounded by C for all points (x, t) in $\bar{\Omega}$.*

Proof Alongside boundary $\partial\bar{\Omega}$, $u = 0$. Hence, $u_t = 0$. Notice that $u(x, 0) = 0$ at initial boundary, which implies $u_x = 0$, $u_{xx} = 0$. $L_\epsilon u(x, t)$ gives

$$u_t(x, 0) = 0, \ 0 \le x \le 1.$$

Therefore, on $\partial\bar{\Omega}$, $u_t = 0$. Applying the same operator on u_t,

$$\mathsf{L}_\epsilon u_t(x,t) = -\epsilon u_{txx}(x,t) + \tilde{b}(\tilde{\mathsf{x}})u_{tx}(x,t) - \tilde{c}(\tilde{\mathsf{x}})u_t(x,t) + u_{tt}(x,t),$$

which equals zero. Choose $\nu(x,t) = C > 0$ as barrier function for u_t. Applying maximum principle, one obtains desired outcome. □

Lemma 9 *The absolute value of* $u_x(x,t)$ *is bounded by* $C\left(1+\epsilon^{-1}e^{\frac{-\Lambda(1-x)}{\epsilon}}\right)$ *for all points* (x,t) *in* $\bar{\Omega}$.

Proof Given a fixed $t \in [0,1]$, this outcome is derived via applying arguments as in [40] along (x,t) such that $x \in [0,1]$, as the only bounds needed are of Lemmas 7 and 8. □

Lemma 10 *The absolute value of* $u_{tt}(x,t)$ *is bounded by* C *for all points* (x,t) *in* $\bar{\Omega}$.

Proof On the boundary $x = 0$ as well as the boundary $x = 1$ along $\bar{\Omega}$, $u_{tt} = 0$. Next, on the line $t = 0$, proceed in similar manner like done previously (refer Lemma 8) and get the desired result. □

Lemma 11 *The absolute value of* $u_{xt}(x,t)$ *is bounded by* $C\left(1+\epsilon^{-1}e^{\frac{-\Lambda(1-x)}{\epsilon}}\right)$ *for all points* (x,t) *in* $\bar{\Omega}$.

Proof Differentiating linearized equation $\mathsf{L}_\epsilon u(x,t)$ w.r.t. t,

$$-\epsilon u_{xxt} + \tilde{b}u_{xt} - \tilde{c}u_t + u_{tt} = 0 \ \ on\ \bar{\Omega}. \tag{5.17}$$

For a fixed $t \in [0,1]$, applying the methodology of Kellogg and Tsan [40] (Lemmas 2.2 and 2.3) on the line segment $\{(x,t) : 0 \le x \le 1\}$, using (5.17) and Lemmas 7, 8 and 10 gives the result. □

Lemma 12 *The absolute value of* $u_{xx}(x,t)$ *is bounded by* $C\left(1+\epsilon^{-2}e^{\frac{-\Lambda(1-x)}{\epsilon}}\right)$ *for all points* (x,t) *in* $\bar{\Omega}$.

Proof Taking the derivative of (5.6) in relation to x,

$$\begin{aligned} -\epsilon u_{xxx} + \tilde{b}u_{xx} - \tilde{c}u_x + u_{tx} &= 0 \ \ on\ \bar{\Omega}. \\ \implies -\epsilon u_{xxx} + \tilde{b}u_{xx} &= \tilde{\mathsf{h}}(x,t) \ \ on\ \bar{\Omega}. \end{aligned}$$

Using Lemmas 9 and 11, we have

$$\left|\tilde{\mathsf{h}}(x,t)\right| \le C\left(1+\epsilon^{-1}e^{\frac{-\Lambda(1-x)}{\epsilon}}\right). \tag{5.18}$$

Next, consider a fixed $t \in [0,1]$, utilizing the methodology as in [40] along $\{(x,t) : x \in [0,1]\}$ and (5.18) gives the required bound on u_{xx}. □

Theorem 13 *The solution* $u(x,t)$ *associated with the governing equation adheres to estimates*

$$\left|\left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial t}\right)^j u(x,t)\right| \le C\left(1+\epsilon^{-1}e^{\frac{-\Lambda(1-x)}{\epsilon}}\right), \ \ \forall\ (x,t) \in \bar{\Omega}. \tag{5.19}$$

with $k,\ j \ge 0$ *under the conditions* $0 \le k \le 2$, $0 \le j \le 2$ *and* $0 \le k+j \le 2$.

Proof Proof is derived as a result of Lemma 4-12. □

5.3 Finite Element Method

This subsection briefly deals with the explanation of the finite element formulation for the continuous problem (2.1). To start with, firstly we discuss the domain discretization strategy in the following subsection.

5.3.1 Shishkin Mesh

Shishkin mesh represents a piecewise uniform mesh in which total count of mesh points are assumed to be $\{N : N \in 2\mathbb{N},\ N \geq 4\}$. Shishkin mesh works by partitioning $\tilde{\Omega}$ as subintervals $[0, \omega]$ and $[\omega, 1]$. Here, the boundary layer width $\omega = min\{0.5, K\epsilon log N\}$, is the transition parameter, such that K denotes a constant that is chosen suitably and does not depend on ϵ or N. Subdivide intervals $[0, 1-\omega]$ and $[1-\omega, 1]$ into $\frac{N}{2}$ equal segments. Consider $\omega \ll 1$, so that the mesh is coarse in the interval $[0, 1-\omega]$ while fine in $[1-\omega, 1]$. Additionally, assigning half of the equal mesh points to each subinterval and denoting mesh width by h such that:

$$h = \begin{cases} h_1 = \frac{2\omega}{N} & k = 1, 2, ..., \frac{N}{2} \\ h_2 = \frac{2(1-\omega)}{N} & k = 1, 2, ..., N. \end{cases}$$

and $\{x_k\}_{k=1}^{N}$, the mesh points are:

$$x_k = \begin{cases} kh_1, & k = 1, 2, ..., \dfrac{N}{2} \\ \omega + \left(k - \frac{N}{2}\right) h_2, & k = \frac{N}{2} + 1, ..., N \end{cases}$$

Hence,

$$\tilde{\Omega}_h = \bigcup_{e=1}^{N} \Omega_h^e, \qquad \Omega_h^e \text{ denotes the element } [x_{e-1}, x_e].$$

5.3.2 Weak Formulation

In this subsection, weak formulation has been discussed in brief. Temporal discretization has been performed prior to spatial discretization and has already been discussed in subsection 5.1.

The spatial discretization of the semi-discretized system is carried out using Petrov-Galerkin finite element method where the standard piecewise continuous linear hat functions have been used for the trial space. We utilize exponentially fitted splines as basis functions for constructing the test space which has been discussed in the next section.

We denote

$$B(w, v) = \int_0^1 (-\epsilon w_x v_x + b w_x v) dx$$

for further computations and v is any test function spanned by the exponentially fitted splines $\vartheta^{i,m}$. Now, using the finite element approximation $y(x, t) = \sum_{i=0}^{N} y_i(t)\Phi^i(x)$, the resulting system yields

$$A^i\{y^i\} = F^i,$$

where the superscript i represents the time level at which the solution has been calculated.

6 Stability and Uniform Convergence

Introduce the approximation

$$\int_0^1 y_t(x,t)\vartheta^j(x,t) \approx h_j y_t(x_i,t), \tag{6.1}$$

Numerical scheme generated via utilizing this approximation is known as mass-lumped numerical scheme. Let the approximation $\bar{b}(x,t)$ for $\tilde{b}(x,t)$ be defined as

$$\bar{b}(x,t) = \tilde{b}(x_i,t), \quad \forall x \in (x_{i-1}, x_i] \text{ for } 1 \le i \le N.$$

We begin by introducing the trial functions $\Phi^i(x)_{i=0}^N$ as standard piecewise linear hat functions, that do not depend on t, specifically

$$\Phi^i_{xx} = 0 \quad \text{on} \quad \bigcup_{i=1}^{N}(x_{i-1}, x_i)$$
$$\text{and} \quad \Phi^i(x_j) = \delta_{ij} \quad \forall\, j = 0, ..., N,$$

with δ_{ij} representing the Kronecker delta.

Next, choose test functions $\left\{\vartheta^{i,m}(x)\right\}_{i=0}^N$ as exponentially fitted $\overline{L}^* - splines$ that serve as solutions of

$$-\epsilon\vartheta^i_{xx}(x,t_m) - b(x,t_{m+1})\vartheta^i_x(x,t_m) = 0 \text{ on } \bigcup_{i=1}^{N}(x_{i-1}, x_i)$$
$$\text{and} \quad \vartheta^i(x_j,t_m) = \delta_{ij} \quad \text{for } j = 0, 1, 2, ..., N, \tag{6.2}$$

where each $\vartheta^{i,m}$ is compactly supported with support (x_{i-1}, x_{i+1}) and $\vartheta^{i,m} = \vartheta^i(x,t_m)$.

Our approach is given by

$$\overline{B}_m(y,\vartheta^{i,m}) + h_i\tilde{c}_{i,m}y_{i,m} + h_i\left(\frac{y_{i,m+1} - y_{i,m}}{k}\right) = 0 \tag{6.3}$$

for i ranging from 0, 1, ..., N, along with m from 0, 1, ..., $M-1$, and

$$\overline{B}_m(w,v) = \int_0^1 \left(\epsilon w_x v_x + \bar{b}(x_i,t_{m+1})w_x v\right)_m dx,$$

where subscript m tells that the computation is being carried out at $t = t_m$.

We evaluate the bilinear form $\overline{B}_m(y,\vartheta^{i,m})$ at y taken as Φ^{i-1}, Φ^i and Φ^{i+1}. A general representation of this equation can be expressed as

$$\begin{aligned} -\epsilon\left[h_i^{-1}\sigma(-\rho_i)(y_{i-1,m} - y_{i,m}) - h_{i+1}^{-1}\sigma(\rho_{i+1})(y_{i,m} - y_{i+1,m})\right] \\ +h_i\tilde{c}_{i,m}y_{i,m} + h_i\left(\frac{y_{i,m+1} - y_{i,m}}{k}\right) = 0, \end{aligned} \tag{6.4}$$

where $\rho_i = \frac{b_{i,m+1}h_i}{\epsilon}$ and $\sigma(x) = \frac{x}{e^x - 1}$.

We can write the scheme as in (6.4) in matrix notation as

$$-\left(A_m + k^{-1}\tilde{I}\right)\overrightarrow{y}_m + \left(k^{-1}\tilde{I} + J\right)\overrightarrow{y}_{m+1} = \overrightarrow{q}_{m+1}, \tag{6.5}$$

where (A_m) denotes tridiagonal matrix of dimension $(N+1)\times(N+1)$ having 0-th as well as N-th rows equal to 0.

Up to $(N-1)^{th}$ entry,

$$\begin{aligned}(A_m)_{i,i-1} &= \epsilon h^{-2}\sigma(-\rho_{i,m}),\\ (A_m)_{i,i+1} &= \epsilon h^{-2}\sigma(\rho_{i+1,m}),\\ (A_m)_{i,i} &= -(A_m)_{i,i-1} - (A_m)_{i,i+1} + \tilde{c}_{i,m}, \quad m = 0, 1, 2, ..., M.\end{aligned}$$

$\tilde{I}_{(N+1)\times(N+1)}$ signifies the identity matrix, where first as well as last rows are 0. $(J)_{(N+1)\times(N+1)}$ can be utilized for including boundary conditions where $(0, 0)$ and (N, N) components are assigned as one, while remaining entries equal to 0. For column vectors, we have used the notations

$$\overrightarrow{y}_m = (y_0, y_1, y_2, ..., y_N)_m^T, \quad \overrightarrow{q}_m = (q_0(t_m), 0, 0, ..., 0, q_1(t_m))^T,$$

having 'T' as transpose.

Here, $k^{-1}\tilde{I} + J$ identifies as diagonal matrix containing positive elements, making it invertible. Therefore, based on (6.5), one obtains

$$\overrightarrow{y}_{m+1} = \left(k^{-1}\tilde{I} + J\right)^{-1} \times \left[\left(A_m + k^{-1}\tilde{I}\right)\overrightarrow{y}_m + \overrightarrow{q}_{m+1}\right]. \tag{6.6}$$

Next, we will find the conditions on h_i and k for which every entry in the tridiagonal matrix $[A_m + k^{-1}\tilde{I}]_{(N+1)\times(N+1)}$ is greater than zero.

$$\begin{aligned}(A_m)_{i,i-1} &= \frac{h_i^{-1} b_{i,m+1}}{1 - e^{\frac{-b_{i,m+1}h_i}{\epsilon}}},\\ (A_m)_{i,i+1} &= \frac{h_{i+1}^{-1} b_{i+1,m+1}}{e^{\frac{b_{i+1,m+1}h_{i+1}}{\epsilon}} - 1},\\ k\,(A_m)_{i,i} &= -k\left[(A_m)_{i,i-1} + (A_m)_{i,i+1} - \tilde{c}_{i,m}\right].\end{aligned}$$

Therefore, we have

$$k(A_m)_{i,i} \geq -k\left[\frac{\alpha^* h_i^{-1}}{1 - e^{\frac{-b_{i,m+1}h}{\epsilon}}} + \frac{\alpha^* h_{i+1}^{-1}}{e^{\frac{b_{i+1,m+1}h}{\epsilon}} - 1}\right].$$

Choose

$$h = \max_{1\leq i\leq N} h_i.$$

Since, $\frac{1}{1-e^{-x}}$ and $\frac{1}{e^x-1}$ exist as decreasing functions for all $x \geq 0$, it is always possible to determine h and k which allows for

$$\left|kh^{-1}\left(\frac{\alpha^*}{1 - e^{\frac{-b_{i,m+1}h}{\epsilon}}} + \frac{\alpha^*}{e^{\frac{b_{i+1,m+1}h}{\epsilon}} - 1}\right)\right| < 1. \tag{6.7}$$

$$\Rightarrow \left(kA_m + \tilde{I}\right)_{i,i} > 0.$$

Now, we define the discrete operator $L_{h,k}$ as

$$(L_{h,k}y)_{i,m+1} \equiv \overline{B}_m\left(y_m, \vartheta^i\right) + h_i\tilde{c}_{i,m}y_{i,m} + h_i\left(\frac{y_{i,m+1} - y_{i,m}}{k}\right). \tag{6.8}$$

Since, by assumption, we have $y_0 \geq 0$, the following theorem is a straight forward consequence of (6.6) and (6.7).

Theorem 14 *Consider $L_{h,k}$ as outlined in (6.8). If h and k can be determined such that they satisfy the inequality in (6.7), then $L_{h,k}$ satisfies a discrete maximum principle.*

Lemma 15 [41, 42] *For a fixed $t \in [0, 1]$, the following inequality holds*

$$\left|\left(1, \vartheta^{i,m}\right) - h_i\right| \leq C h_i^2.$$

Theorem 16 *Let the coefficients for linearized problem be sufficiently smooth to ensure that the estimates in (5.19) holds and the stability condition in (6.7) is satisfied, then*

$$\left|u\left(x_i, t_m\right) - y_{i,m}\right| \leq C\left(h + k\right) \quad \forall\, i, m.$$

Proof We shall prove the result for a fixed i ranging from 1 to $N-1$, along with m ranging from 1 to $M-1$. Assume $e_{i,m} = u(x_i, t_m) - y_{i,m}$.

Consider the operator

$$(L_{h,k}v)_{i,m} = h_i^{-1}\left\{\overline{B}\left(v, \vartheta^{i,m}\right)_{i,m} + h_i \tilde{c}_{i,m} v_{i,m} + h_i\left(\frac{v_{i,m+1} - v_{i,m}}{k}\right)\right\}.$$

$$\begin{aligned}
\Rightarrow h_i(L_{h,k}e)_{i,m} &= \overline{B}\left(u, \vartheta^{i,m}\right)_{i,m} + h_i \tilde{c}_{i,m} u_{i,m} + h_i\left(\frac{u_{i,m+1} - u_{i,m}}{k}\right)\\
&= \left\{\left(\left(\overline{b}_{i,m+1} - b(x, t_{m+1})\right) u_x, \vartheta^i\right)_{i,m} - (\tilde{c}u, \vartheta^i)_{i,m} + h_i \tilde{c}_{i,m} u_{i,m}\right\}\\
&\quad - \left(u_t, \vartheta^{i,m}\right)_{i,m} + h_i\,(u_t)_{i,m} + O\left(h_i k^3\right)\\
&\leq \left\{\left(\left(\overline{b}_{i,m+1} - b(x, t_{m+1})\right) u_x - \tilde{c}_m u + \tilde{c}_{i,m} u_{i,m} - u_t + (u_t)_{i,m}, \vartheta^i\right)_{i,m}\right\}\\
&\quad + O\left(h_i^2\right) + O\left(h_i k\right). \qquad \text{(using Lemma (15))}.
\end{aligned}$$

Using Theorem 13, the above inequality can be written as

$$\Rightarrow (L_{h,k}e)_{i,m} \leq C\int_{x_{i-1}}^{x_{i+1}} \epsilon^{-1} e^{\frac{-\Lambda(1-x)}{\epsilon}}\,dx + Ch_i + Ck.$$

A simple integration reduces this inequality to

$$\Rightarrow (L_{h,k}e)_{i,m} \leq Ce^{\frac{-\Lambda(1-x_{i+1})}{\epsilon}}\left(1 - e^{-2\rho_i}\right) + Ch_i + Ck, \qquad \rho_i = \frac{\Lambda h_i}{\epsilon}.$$

Choose a barrier function

$$w = Ch_i\left(1 + e^{-\frac{\rho_i}{2}}\right)^2 e^{\frac{-\Lambda(1-x)}{2\epsilon}} + C\left(h_i + k\right)x \quad \text{on } \overline{\Omega}.$$

On boundary points $x = 0, 1$ and $t = 0$, $|e| \leq |w|$.

Again, operating $L_{h,k}$ on w gives

$$\begin{aligned}
\left(L_{h,k}w\right)_{i,m} &= h_i^{-1}\{\overline{B}\left(w, \vartheta^{i,m}\right)_{i,m} + h_i \tilde{c}_{i,m} w_{i,m}\}\\
&\geq h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(-\epsilon w_{xx}(x) + \overline{b}(x_i, t_{m+1}) w_x(x)\right)\vartheta^{i,m}(x)\,dx.
\end{aligned}$$

Since,

$$-\epsilon w_{xx}(x) + \overline{b}_{i,m+1} w_x(x) \geq C\Lambda h_i \epsilon^{-1}\left(1 + e^{\frac{-\rho_i}{2}}\right)^2 e^{\frac{-\Lambda(1-x)}{2\epsilon}} + C\left(h_i + k\right),$$

$$\Rightarrow (L_{h,k}w)_{i,m} \geq \int_{x_{i-1}}^{x_{i+1}} \left(C\epsilon^{-1} \left(1 + e^{\frac{-\rho_i}{2}}\right)^2 e^{\frac{-\Lambda(1-x)}{2\epsilon}} + C\left(h_i + k\right) \right) \vartheta^{i,m} dx$$
$$\geq C\left(1 - e^{-2\rho_i}\right) e^{-\frac{-\Lambda(1-x_{i+1})}{\epsilon}} + C\left(h_i + k\right)$$
$$\Rightarrow (L_{h,k}w)_{i,m} \geq (L_{h,k}e)_{i,m}.$$

Using the discrete maximum principle property of the operator $L_{h,k}$, we get

$$|e_{i,m}| \leq w_{i,m}$$
$$\text{i.e. } |u(x_i, t_m) - y_{i,m}| \leq Ch_i \left(1 + e^{-\frac{\rho_i}{2}}\right)^2 e^{\frac{-\Lambda(1-x)}{2\epsilon}} + C\left(h_i + k\right) x$$
$$\leq Ch_i + C\left(h_i + k\right)$$
$$= Ch_i + Ck.$$

Choosing $h = \max_{1 \leq i \leq N} \{h_i\}$ and the generic constant C to be large enough, we get

$$|u(x_i, t_m) - y_{i,m}| \leq Ch + Ck \quad \forall\, i, m.$$

□

7 Comparative Analysis of LSA and FEM

Herein, we present comparative analysis of solutions obtained through the Lie symmetry method and Finite element method. By solving SPGBHE using both techniques, we assess the accuracy of the FE and LS solutions through an error analysis at selected time points. In particular, the LS solution

$$\left[\frac{\gamma}{2} + \frac{\gamma}{2} \tanh\left(\frac{\delta\gamma\left(\alpha + \mathcal{Y}\right)\left(-t\left(\delta - \gamma + 1\right)\mathcal{Y} + t\alpha\left(1 + \gamma + \delta\right) - 2x\left(\delta + 1\right)\right)}{16\epsilon\left(\delta + 1\right)^2} \right)\right]^{\frac{1}{\delta}}$$

is examined alongside the FE solution for initial and boundary conditions are taken as:

$$u(x, 0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} \tanh\left(-\frac{\delta\gamma\left(\alpha + \mathcal{Y}\right)x}{8\epsilon\left(\delta + 1\right)} \right)\right]^{\frac{1}{\delta}},$$
$$u(0, t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} \tanh\left(\frac{\delta\gamma\left(\alpha + \mathcal{Y}\right)\left(-t\left(\delta - \gamma + 1\right)\mathcal{Y} + t\alpha\left(1 + \gamma + \delta\right)\right)}{16\epsilon\left(\delta + 1\right)^2} \right)\right]^{\frac{1}{\delta}},$$
$$u(1, t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} \tanh\left(\frac{\delta\gamma\left(\alpha + \mathcal{Y}\right)\left(-t\left(\delta - \gamma + 1\right)\mathcal{Y} + t\alpha\left(1 + \gamma + \delta\right) - 2\delta - 2\right)}{16\epsilon\left(\delta + 1\right)^2} \right)\right]^{\frac{1}{\delta}}.$$

. with $\mathcal{Y} = \sqrt{\alpha^2 + 4\epsilon\beta\left(\delta + 1\right)}$. The influence of varying singular perturbation parameter (ϵ) on the reliability of the solution obtained through each procedure is examined in the error table below (Table 2). This analysis explores a range of perturbation values from 10^{-2} to 10^{-10} across different time levels. Noticeably, at higher perturbations ($\epsilon = 10^{-2}$), the errors are more significant.

To provide more insights into the reliability and accuracy of the techniques, we provide plots illustrating the solution profiles obtained via FEM and LSA. The overlapping of solutions in Figs. 1 and 2 demonstrates that the results provided by FEM are consistent with LSA and vice-versa. Because of the singular perturbation parameter effect, Fig. 2 (with

Table 2 L_∞ error analysis of FE and LS solutions: Illustration of errors at different perturbation values corresponding to varying time levels with parametric assumptions: $\gamma = 0.001, \delta = 1, \alpha = 1, \beta = 1$

t	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-8}$	$\epsilon = 10^{-10}$
0.1	1.25204E-008	1.49852E-008	5.49054E-007	5.68905E-005	9.96783E-004
0.2	2.49815E-008	2.99700E-008	1.097159E-006	1.02749E-004	9.96783E-004
0.3	3.75020E-008	4.49545E-008	1.91716E-006	1.47679E-004	9.96783E-004
0.4	4.99631E-008	5.99386E-008	2.46215E-006	1.81407E-004	9.96783E-004
0.5	6.24836E-008	8.24140E-008	3.00557E-006	2.10156E-004	9.96783E-004
0.6	7.49446E-008	9.73971E-008	3.54726E-006	2.29231E-004	9.96783E-004
0.7	8.74648E-008	1.12379E-007	4.35613E-006	2.45985E-004	9.96783E-004
0.8	9.99248E-008	1.27362E-007	4.89267E-006	2.58741E-004	9.96783E-004
0.9	1.12442E-007	1.42344E-007	5.42681E-006	2.71604E-004	9.96783E-004
1.0	1.24955E-007	1.57325E-007	5.95837E-006	2.81216E-004	9.96783E-004

$\epsilon = 10^{-8}$) demonstrates a stronger boundary layer formation as compared to Fig. 1 (with $\epsilon = 10^{-6}$). This indicates that Fig. 2 exhibits a more pronounced boundary layer effect. It is concluded that FE and LS solutions show good agreement, further reinforcing that both methods accurately capture the boundary layer effect in singularly perturbed problems. This further validates the authenticity of FEM and LSA. Similar results have been noticed for other approximate analytical Lie symmetry solutions when compared with those of exponentially fitted FEM.

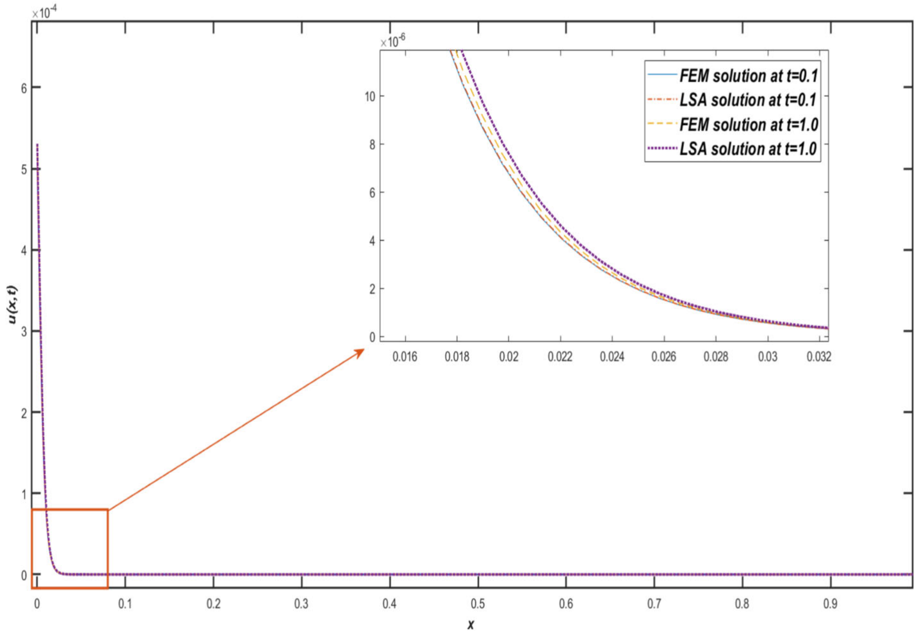


Fig. 1 Plot illustrating FE and LS solutions having singular perturbation value, $\epsilon = 10^{-6}$ with parametric assumptions $\gamma = 0.001, \delta = 1, \alpha = 1, \beta = 1$ at time levels 0.1 and 1.0

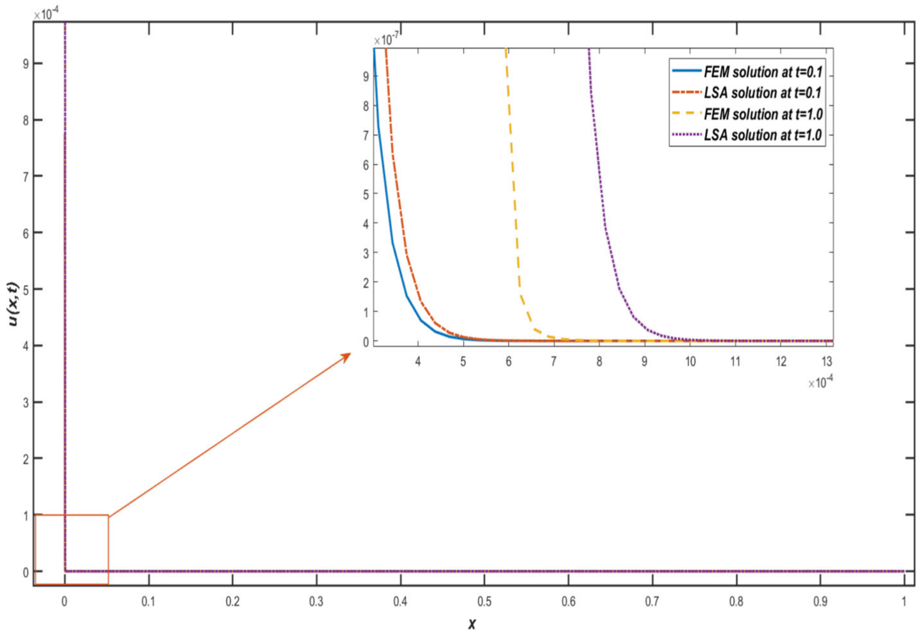
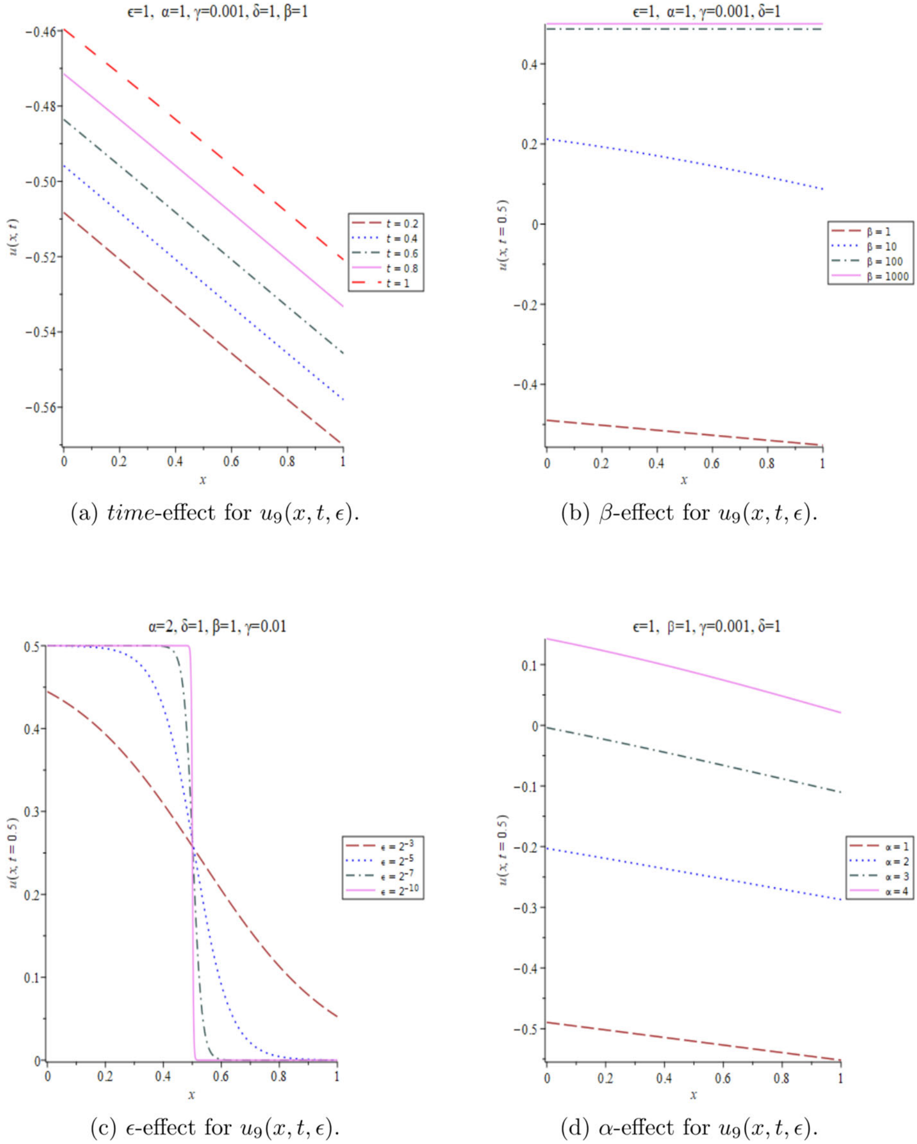


Fig. 2 Plot illustrating FE and LS solutions having singular perturbation value, $\epsilon = 10^{-8}$ with parametric assumptions $\gamma = 0.001$, $\delta = 1$, $\alpha = 1$, $\beta = 1$ at time levels 0.1 and 1.0

8 Graphical Representations of Exact Solutions

Singularly perturbed problems necessitate graphic interpretations in order to enhance the understanding of perturbation effects, envision boundary layers and rapid transitions comprehending the solution behaviors. Since the governing equation possess four additional parameters as well, hence, visualization of the impact of parameter transitions on the solution is also crucial. Herein, we illustrate several solutions by providing plots. Figure 3(a) depicts the temporal behavior of the exact solution $u_9(x, t, \epsilon)$. It suggests a linear and consistent decrease over time with rate of decrease being uniform. Sub-figure (b) indicates that greater value of β results in solution being more stable. The singular perturbation parameter effect demonstrated by Fig. 3(c) suggests that as ϵ gets smaller, the solution rapidly decreases near the boundary. An abrupt change in the pattern of transition from high to low values is observed, from a gradual decline ($\epsilon = 2^{-3}$) to a sharp drop off ($\epsilon = 2^{-10}$). Figure 3(d) notices the influence of α. Analyzing the solution profiles of the exact solution $u_4(x, t, \epsilon)$ from Fig. 4, one can notice in (a) that for $\delta = 1$, it is approximately zero throughout the domain, $\delta = 2$ indicates that function stabilizes at a value ≈ 0.068. For $\delta = 3$, it initiates from a high value of around 0.16 at origin and decreases linearly towards ≈ 0.11 as x approaches 1. Figure 4(b) depicts for $\epsilon = 2^{-8}$, function declines at the slowest rate, as ϵ decreases, ($\epsilon = 2^{-10}, 2^{-12}, 2^{-14}$), the decline is rapid.

Surface plots in Fig. 5(a) presents a smoother transition around origin at $\epsilon = 10^{-2}$, (b) presents the same solution profile with smaller ϵ value which has very steep slopes, observing a more abrupt change for $u_4(x, t, \epsilon)$.



(a) *time*-effect for $u_9(x, t, \epsilon)$.

(b) β-effect for $u_9(x, t, \epsilon)$.

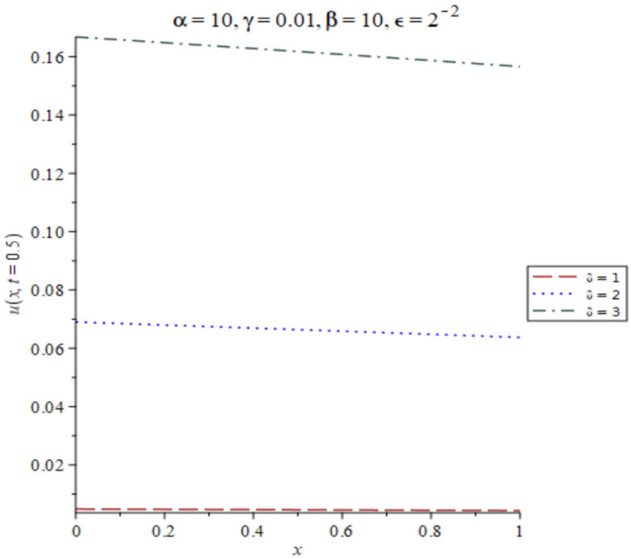
(c) ϵ-effect for $u_9(x, t, \epsilon)$.

(d) α-effect for $u_9(x, t, \epsilon)$.

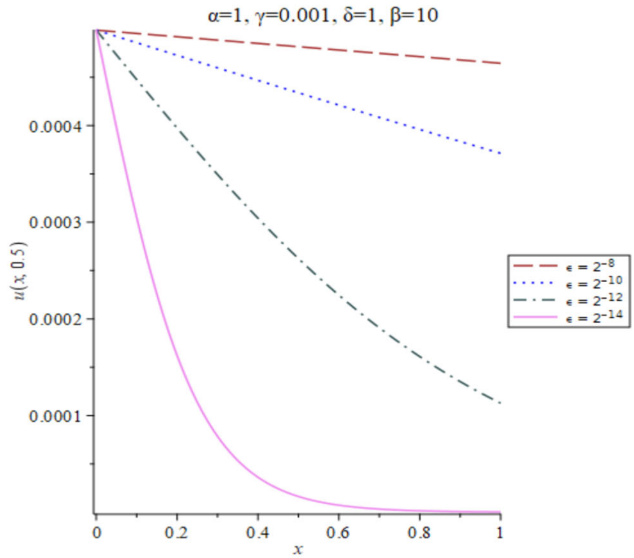
Fig. 3 Solution profiles of (4.3) for its exact solution $u_9(x, t, \epsilon)$

To provide the visualization of parameter effects on the exact solutions, we analyze Fig. 6 which illustrates two of the exact solutions ($u_5(x, t, \epsilon), u_7(x, t, \epsilon)$) within space-time domain. Very sharp boundary layers can clearly be observed.

In snippet, it is easily demonstrated that the formation of boundary layers is not solely due to ϵ and that other parameters also play a crucial role in affecting the solution profiles. The authors have observed similar behavior for the other exact solutions mentioned in the study.

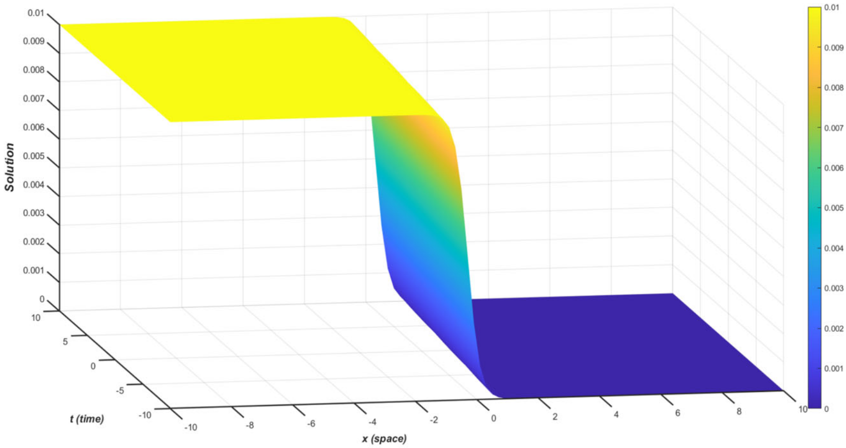


(a) δ-effect.

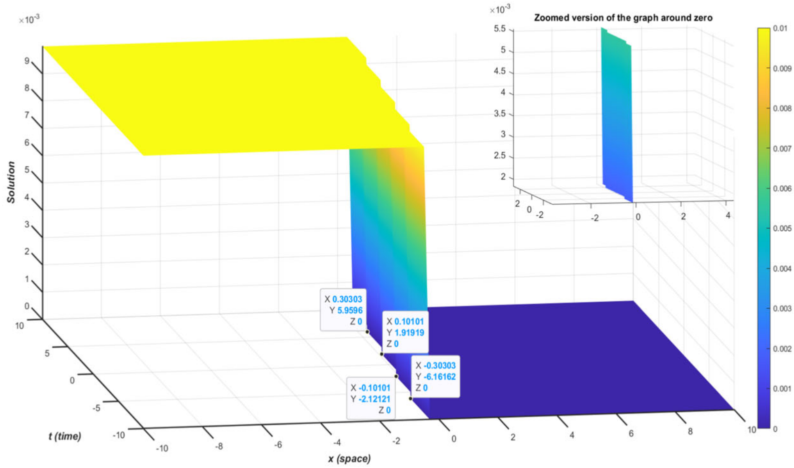


(b) ϵ-effect.

Fig. 4 Solution profiles for the exact solution $u_4(x, t, \epsilon)$ corresponding to (4.3) of reduction 3



(a) $\epsilon = 10^{-2}$.

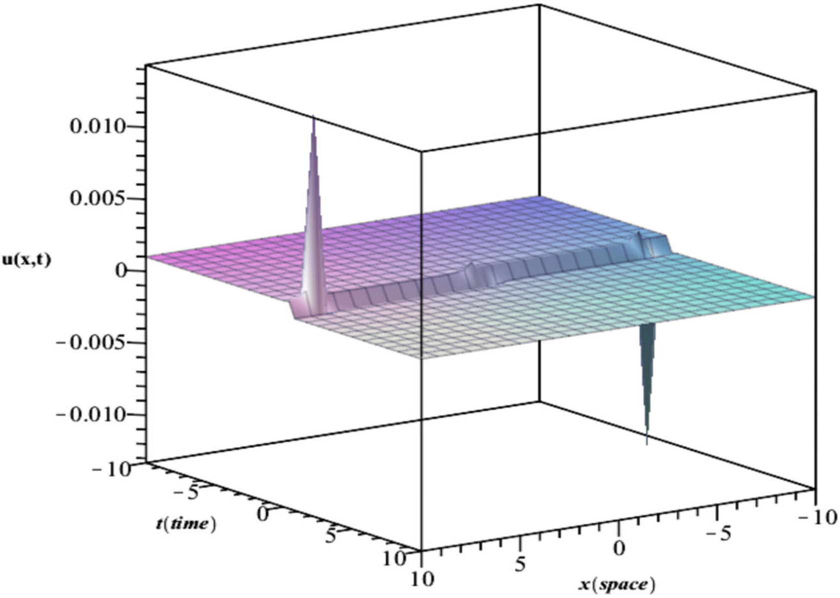


(b) $\epsilon = 10^{-10}$

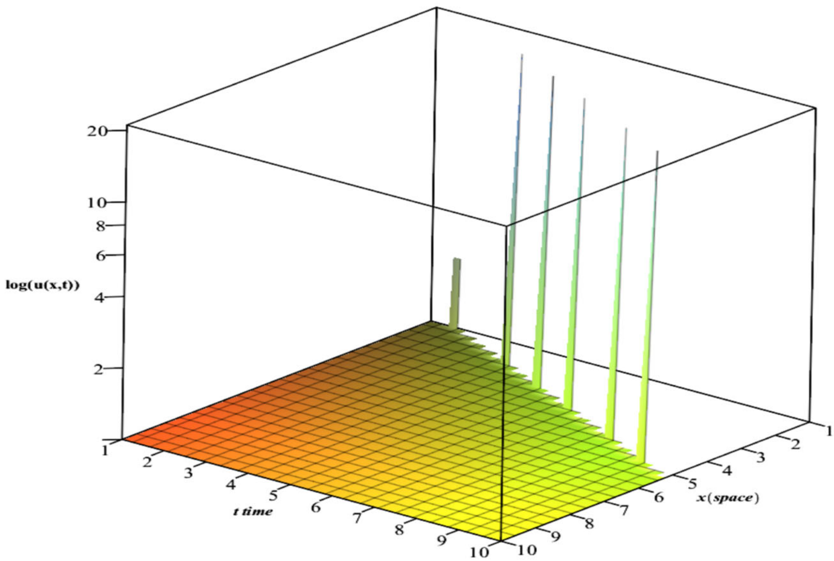
Fig. 5 Surface plot at parameter values $\delta = 1,\ \beta = 10,\ \alpha = 10,\ \gamma = 0.01$ of (4.3) for $u_4(x, t, \epsilon)$

9 Conclusion of the Study

The study has been focused on the nonlinear SPGBHE. Approximate LSA and EF-FEM are two approaches applied to derive the analytical and approximate solutions to the problem under consideration. By employing BGI method within the framework of approximate LSA, a five-dimensional Lie algebra has been established, leading to symmetry reductions. Subsequently, the well-established tanh-coth method has been applied to solve some of the



(a) $u_5(x, t, \epsilon)$.



(b) $u_7(x, t, \epsilon)$.

Fig. 6 Solution profiles for some exact solutions of (4.3) assigning fixed parameter values $\delta = 1,\ \alpha = 100,\ \beta = 100,\ \epsilon = 10^{-20}\ \gamma = 0.001$

obtained reductions. It is analyzed that symmetry analysis is a robust and effective method for obtaining solutions to otherwise challenging nonlinear PDEs. Moreover, a thorough stability and uniform convergence analysis of the proposed numerical scheme has been conducted, ensuring reliability of the results. A comparison of numerical solutions derived via EF-FEM with the approximate solution derived from LSA revealed a strong agreement, highlighting the efficacy of the two methods. The error table and solution plots for varying perturbation values further illustrate the robustness of both the schemes in handling the governing equation.

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Declarations

Competing Interests The authors declare that they have no conflict of interest.

Competing Interests The authors declare no competing interests.

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