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Jagjeet Jakhar
Central University of Haryana

Renu Chugh
Maharshi Dayanand University

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FUZZY STABILITY OF MIXED TYPE FUNCTIONAL EQUATIONS IN MODULAR SPACES

Jagjeet Jakhar[1], Jyotsana Jakhar[*2] and Renu Chugh[2]

[1]Department of Mathematics, Central University of Haryana, Mahendergarh-123031, INDIA

[2]Department of Mathematics, M.D. University, Rohtak-124001, Haryana, INDIA

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Abstract. In this article, our aim is to find some stability results for mixed type cubic and quartic functional equations in fuzzy modular spaces using the fundamental results of fixed-point theory. The fixed point method provides one of the effective techniques that can be used to investigate the fuzzy stability of a mixed type cubic and quartic functional equations.

1. **Introduction.** Applications of “stability theory of functional equations” for the proof of ”new fixed point theorems” with applications were provided by Isac and Rassias [7] in 1996. Number of authors have investigated the stability problems of several functional equations using fixed point method (see [2, 3, 21, 22, 35]).

Mixed type functional equation is defined as “single functional equation has more than one behaviour”. The researchers in the field of mixed type functional equations have obtained only few functional equations (see [5, 8, 14, 13, 22, 25, 26, 30, 33, 34]).

In 1940, Ulam [29] raised a question “Is there an exact homomorphism close to approximate homomorphism?” and established the first stability problem. Hyers [6] in Banach spaces provided the answer to this problem. Nakano [19] studied the theory of modular linear spaces in 1950, since then this theory have been thoroughly established by many authors, referred as Amemiya [1], Koshi [9], Luxemburg [15], Mazur [17], Musielak [18], Orlicz [20], Rassias [24, 25], Turpin [28] and Yamamu [34]. The theory of modular spaces is largely applicable in the Orlicz spaces [20] and the theory of interpolation [9, 12]. The concept of fuzzy modular spaces was introduced by Shen and Chen [27] in 2013. Kumam [11] and Wongkum et al. [31] then introduced the fixed point concept in fuzzy modular spaces and obtained some of its properties. Hyers-Ulam stability of sextic functional equation in fuzzy modular spaces was scrutinized by Wongkum and Kumam [32]. Based on the notion of fuzzy modular spaces, we investigate the general solution and Hyers-Ulam stability of mixed type cubic and quartic functional equation

$$g(u+2v)+g(u-2v)=4(g(u+v)+g(u-v))-24g(v)-6g(u)+3g(2v) \qquad (1)$$

in fuzzy modular spaces which were introduced by Gordji et al. [11]. It is easy to see that the function $g(u)=au^3+bu^4$ is a solution of the functional equation (1).

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*Corresponding author: Jyotsana Jakhar.

Primarily, we are going to introduce the definitions, notations and usual properties of the given spaces.

Definition 1.1. [12] "Let T be a linear space over a field $\mathbb{K}$ ($\mathbb{R}$ *or* $\mathbb{C}$). For any $u, v \in T$, we says that a generalized functional $\mu : T \to [0, \infty)$ is a modular if μ satisfies
(M_1) $\mu(u) = 0$ if and only if $u = 0$,
(M_2) $\mu(au) = \mu(u)$ for all scalars a with $|a| = 1$,
(M_3) $\mu(au + bv) \leq \mu(u) + \mu(v)$ for all scalars $a, b \geq 0$ with $a + b = 1$.
If (M_3) is replaced by
(M_4) $\mu(au + bv) \leq a\mu(u) + b\mu(v)$ for all scalars $a, b \geq 0$ with $a + b = 1$, then functional μ is called a convex modular".

"A modular μ defines the following vector space

$$T_\mu = \{u \in T : \mu(au) \to 0 \ as \ a \to 0\}$$

and T_μ is a modular space".

Definition 1.2. [27](Fuzzy modular space) Let μ be a fuzzy set on $S \times R^+$, T be a complex or real vector space,α be a zero on T and $*$ be a continuous triangular norm. The triple $(T, \mu, *)$ is said to be a fuzzy modular space and μ is said to be a fuzzy modular if it satisfies
(FM_1) $\mu(u, t') > 0$
(FM_2) $\mu(u, t') = 1$ if and only if $u = \alpha$
(FM_3) $\mu(u, t') = \mu(-u, t')$
(FM_4) $\mu(au + bv, t + t') \geq \mu(u, t) * \mu(v, t'), a, b \geq 0, a + b = 1$
(FM_5) the function $\mu(u, .) : (0, \infty) \to (0, 1]$ is continuous.

Example 1.3. Let μ be a Fuzzy set on $S \times R^+$, T be a complex or real vector space and $*$ be a continuous triangular norm in such a way that $a * b = a *_M b = min\{a, b\}$. Then $(T, \mu, *)$ is a Fuzzy modular space.

Definition 1.4. [27] "Let $(T, \mu, *)$ be a Fuzzy modular space and $\{u_\alpha\}$ be a sequence in T. Then:
(1) $\{u_\alpha\}$ is $\mu-$convergent to a point u and write $u_\alpha \to u$ if $\rho(u_\alpha - u) \to 0 \ as \ \alpha \to \infty$.
(2) $\{u_\alpha\}$ is called Cauchy if for any $\varepsilon > 0$, one has $\rho(u_\alpha - u_\beta) < \varepsilon$ for sufficiently large $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.
(3) Every $\mu-$ convergent sequence in an $FM-$ space is a $\mu-$ Cauchy sequence. In $(T, \mu, *)$, if each $\mu-$ Cauchy sequence is $\mu-$ convergent sequence , then $(T, \mu, *)$ is called a $\mu-$ complete fuzzy modular space."

Definition 1.5. [27] If μ fulfills the property $\mu(\alpha u, t') = \mu(u, \frac{t'}{|\alpha|^b})$ for some fixed $b \in (0, 1]$ and a nonzero real number α, then $(T, \mu, *)$ is said to be a $b-$homogeneous fuzzy modular space.

The definitions of cubic and quartic functional equations which are used by many authors in our research. The function $f(x) = x^3$ satisfies the functional equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$, which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. The function $f(x) = x^4$ satisfies the functional equation $f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x)$, which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In this article, we establish the stability results of mixed type cubic and quartic functional equations which are explored forms of Wongkum et al. [31] by using the fixed point method.

2. **General solution.** Throughout in this section S and T be real vector spaces. In this section, we first prove two lemmas for the proof of our main theorem.

Lemma 2.1. *If an even function $g : S \to T$ satisfies (1), then g is quartic.*

Proof. Substituting $u = v = 0$ in (1), we obtain $g(0) = 0$. Substituting $u = 0$ in (1), evenness of g implies

$$g(2v) = 16g(v) \tag{2}$$

for all $v \in S$. So (1) can be written as

$$g(u + 2v) + g(u - 2v) = 4(g(u + v) + g(u - v)) + 24g(v) - 6g(u). \tag{3}$$

This shows that g is quartic function, which completes the proof of the theorem. □

Lemma 2.2. *If an odd function $g : S \to T$ satisfies (1), then g is cubic.*

Proof. Substituting $u = v = 0$ in (1), we obtain $g(0) = 0$. Substituting $u = 0$ in (1), oddness of g implies

$$g(2v) = 8g(v) \tag{4}$$

for all $v \in S$. So (1) can be written as

$$g(u + 2v) + g(u - 2v) = 4(g(u + v) + g(u - v)) + 24g(v) - 6g(u). \tag{5}$$

Replacing u by $u + v$ in (5), we have

$$g(u + 3v) + g(u - v) = 4g(u + 2v) - 6g(u + v) + 4g(u). \tag{6}$$

Substituting $-v = v$ in (6) implies

$$g(u - 3v) + g(u + v) = 4g(u - 2v) - 6g(u - v) + 4g(u). \tag{7}$$

Now, we subtract (6) from (7) and get

$$g(u + 3v) - g(u - 3v) = 4g(u + 2v) - 4g(u - 2v) - 5g(u + v) + 5g(u - v). \tag{8}$$

If we interchange u and v in (8) then we have

$$g(3u + v) + g(3u - v) = 4g(2u + v) + 4g(2u - v) - 5g(u + v) - 5g(u - v). \tag{9}$$

And replacing $v = u + v$ in (5), we obtain

$$g(3u + 2v) - g(u + 2v) = 4g(2u + v) - 4g(v) - 6g(u). \tag{10}$$

In (10), if we put $v = -v$ then we have

$$g(3u - 2v) - g(u - 2v) = 4g(2u - v) + 4g(v) - 6g(u). \tag{11}$$

Now, add (10) and (11), we obtain

$$\begin{aligned} &g(3u + 2v) + g(3u - 2v) = \\ &4g(2u + v) + 4g(2u - v) + g(u + 2v) + g(u - 2v) - 12g(u). \end{aligned} \tag{12}$$

Setting $u = 2u$ in (8) and using (4), we obtain

$$\begin{aligned} &g(2u + 3v) - g(2u - 3v) = \\ &32g(u + v) - 32g(u - v) - 5g(2u + v) + 5g(2u - v). \end{aligned} \tag{13}$$

Interchanging u and v in (13), gives

$$g(3u+2v)+g(3u-2v)=$$
$$32g(u+v)+32g(u-v)-5g(u+2v)-5g(u-2v). \tag{14}$$

If we compare (12) and (14) and employ (5), we conclude that

$$g(2u+v)+g(2u-v)=2g(u+v)+2g(u-v)+12g(u). \tag{15}$$

This shows that g is cubic function, which completes the proof of the theorem. □

Theorem 2.3. *A mapping $g: S \to T$ satisfies (1) for all $u, v \in S$ iff there exists only a function $G: S \times S \times S \to T$ and a unique symmetric multiadditive function $G': S \times S \times S \times S \to T$ in such a way that $g(u) = G(u,u,u) + G'(u,u,u,u)$ for all $u \in S$ and that G is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Assume g satisfies (1). We break g into even and odd part by putting

$$g_e(u)=\frac{1}{2}(g(u)+g(-u)) \textit{ and } g_o(u)=\frac{1}{2}(g(u)-g(-u)) \tag{16}$$

for all $u \in S$. By (1), we get

$$g_e(u+2v)+g_e(u-2v)=$$
$$\frac{1}{2}[g(u+2v)+g(-u-2v)+g(u-2v)+g(-u+2v)]=$$
$$4(g_e(u+v)+g_e(u-v))-24g_e(v)-6g_e(u)+3g_e(2v) \tag{17}$$

for all $u, v \in S$. This shows that g_e satisfies (1). Similarly we can prove that g_o satisfies (1). By lemmas (2) and (3), g_e and g_o are quartic and cubic, respectively. Thus there exists a unique mapping $G: S \times S \times S \to T$ and a unique symmetric multiadditive function $G': S \times S \times S \times S \to T$ in such a way that $g_e = G'(u,u,u,u)$ and that $g_o = G(u,u,u)$ for all $u \in S$ and G is symmetric for each fixed one variable and is additive for fixed two variables. Thus $g(u) = G(u,u,u) + G'(u,u,u,u)$ for all $u \in S$. The proof of the converse is trivial. □

3. **Stability of the mixed type cubic and quartic functional equations in fuzzy modular spaces.** In this section, we prove the stability of the mixed type functional equations in fuzzy modular spaces for odd case by using fixed point method. For a mapping $g: S \to (T, \varrho, *)$, consider

$$D_o(u,v)=g(u+2v)+g(u-2v)-4(g(u+v)-g(u-v))+24g(v)+6g(u)-3g(2v)$$

Theorem 3.1. *Let S be a vector space, T be a real linear space, $(T, \mu, *)$ be a μ-complete b-homogeneous fuzzy modular space and $\gamma \in \{-1, 1\}$ be fixed. Assume that an odd mapping $g: S \to (T, \varrho, *)$ satisfies*

$$\mu(D_o(u,v),t') \geq \varrho(u,v,t') \tag{18}$$

for all $u, v \in S$ and a given function $\varrho: S \times S \to \triangle$ in such a way that

$$\varrho(2^c u, 2^c v, 2^{3bc} L t') \geq \varrho(u,v,t') \tag{19}$$

and

$$lim_{l\to\infty}\varrho(2^{cl}u,2^{cl}v,2^{3bcl}t')=1 \tag{20}$$

for all $u, v \in S$ and a constant $0 < L < \frac{1}{2^{3b}}$. Then there exists a unique cubic mapping $G : S \to (T, \mu)$ satisfying (1) and

$$\varrho\big(G(u) - g(v), \frac{t'}{2^{3b} L^{\frac{c-1}{2}(1-2^b L)}}\big) \geq \varrho(0, v, t'). \tag{21}$$

Proof. Setting $u = 0$ in (18), we have

$$\mu(3g(2v) - 24g(v), t') \geq \varrho(0, v, t') \tag{22}$$

so

$$\mu(\frac{g(2v)}{2^3} - g(v), t') \geq \varrho(0, v, 2^{3b} t'). \tag{23}$$

Changing v with $2^{-1}v$ in (23), we get

$$\begin{aligned}&\mu(\frac{g(2v)}{2^3} - g(2^{-1}v), \frac{t'}{2^{3b}})\\ &\geq \varrho(0, 2^{-1}v, 2^{3b} 2^b L L^{-1} \frac{t'}{2^{3b}}) \geq \varrho(0, v, 2^{3b} L^{-1} t').\end{aligned} \tag{24}$$

Now from (23) and (24), we get

$$\mu(\frac{g(2^c v)}{2^{3c}} - g(v), t') \geq \Phi(v, t') = \varrho(0, v, 2^{2c} L^{\frac{c-1}{2}} t' 2^b). \tag{25}$$

Assume $R = \{f : S \to (T, \mu) : f(0) = 0\}$ and define η on R as

$$\eta(f) = inf\{m > 0 : \varrho(f(v), mt') \geq \Phi(v, t')\}.$$

It is easy to prove that η is modular on L and includes the $\triangle_2-$condition with $2^b = k$ and Fatou's property. Moreover, L is $\eta-$complete(see[27]).

Let $P : R_\eta \to R_\eta$ be a function and defined as

$$P(G(v)) = \frac{G(2^c v)}{2^{3c}} \ \forall G \in R_\eta.$$

Let $j, k \in R_\eta$ and $m > 0$ be an arbitrary constant with $\eta(j - k) \leq m$. From the definition of η, we have $\mu(f(v) - k(v), mt') \geq \Phi(v, t')$ and then

$$\begin{aligned}\mu(Pf(v) - Pk(v), Lmt') &= \mu(2^{-3c} f(2^c v) - 2^{-3c} k(2^c v), Lmt')\\ &= \mu(f(2^c v) - k(2^c v), 2^{3bc} Lmt')\\ &\geq \Phi(2^c v, 2^{3bc} Lmt')) \geq \Phi(v, t').\end{aligned}$$

Hence $\eta(Pf - Pk) \leq L\eta(j - k)$, this implies that P is an $\eta-$strict contradiction. Changing v by $2^c v$ in (25), we get

$$\mu\big(\frac{g(2^{2c} v)}{2^{3c}} - g(2^c v), t'\big) \geq \Phi(2^c v, t') \tag{26}$$

and therefore

$$\begin{aligned}\mu\big(2^{-2(3c)} g(2^{2c} v) - 2^{-3c} g(2^c v), Lt'\big) &= \big(2^{-3c} g(2^{2c} v) - g(2^c v), 2^{3bc} Lt'\big)\\ &\geq \Phi(2^c v, 2^{3bc} Lt') \geq \Phi(v, t').\end{aligned} \tag{27}$$

Now

$$\begin{aligned}&\mu\big(\frac{g(2^{2c} v)}{2^{2(3c)}} - g(v), 2^b(Lt' + t')\big)\\ \geq\ &\mu\big(\frac{g(2^{2c} v)}{2^{2(3c)}} - \frac{g(2^c v)}{2^{3c}}, Lt'\big) \wedge \mu\big(\frac{g(2^c v)}{2^{3c}} - g(v), t'\big)\\ \geq\ &\Phi(v, t').\end{aligned} \tag{28}$$

In (28) changing v by $2^c v$ and $2^b(Lt'+t')$ by $2^{b(1+3c)}(L^4t'+L^3t')$, we obtain

$$\begin{aligned} & \mu\big(\frac{g(2^{2c}v)}{2^{2(3c)}}-g(2^c v), 2^{3bc}2^b(L^4t'+L^3t')\big) \\ \geq\ & \Phi(2^{2c}v, 2^{3bc}Lt') \geq \Phi(v,t'). \end{aligned} \tag{29}$$

Therefore

$$\mu\big(\frac{g(2^{2c}v)}{2^{3(3c)}}-\frac{g(2^c v)}{2^{3c}}\big)(2^c(L^4t'+L^3t')) \geq \Phi(v,t'). \tag{30}$$

This means

$$\begin{aligned} & \mu\big(\frac{g(2^{3c}v)}{2^{3(3c)}}-g(v), 2^b(2^b(L^4t'+L^3t')+t')\big) \\ \geq\ & \mu\big(\frac{g(2^{3c}v)}{2^{3(3c)}}-\frac{g(2^c v)}{2^{3c}}, 2^b(L^4t'+L^3t'))\big) \wedge \mu\big(\frac{g(2^c v)}{2^{3c}}-g(v), t'\big) \\ \geq\ & \Phi(v,t'). \end{aligned} \tag{31}$$

Now, we generalizing the above inequality, we have

$$\mu\big(\frac{g(2^{cl}v)}{2^{3(cl)}}-g(v), (2^bL)^{l-1}+2^b\sum_{i=1}^{l-1}(2^bL^{i-1}t') \geq \Phi(v,t') \tag{32}$$

where l is a positive integer. So we have

$$\begin{aligned} \eta(P^l g - g) &\leq (2^bL)^{l-1}+2^b\sum_{i=1}^{l-1}(2^bL)^{i-1} \\ &\leq 2^b\sum_{i=1}^{l-1}(2^bL)^{i-1} \\ &\leq \frac{2^b}{1-2^bL}. \end{aligned} \tag{33}$$

Now, It is easy to prove that $\{P^l g\}$ is $\eta-$convergent to $G \in R_\eta$ (see[27]). Therefore (34) becomes

$$\eta(G-g) \leq \frac{2^b}{1-2^bL} \tag{34}$$

this means

$$\mu(G(v)-g(v), \frac{2^b}{1-2^bL}t' \geq \Phi(v,t')) = \varrho(0,v,2^4bL^{\frac{c-1}{2}}t') \tag{35}$$

and so,

$$\mu\big(G(v)-g(v), \frac{t'}{2^{3b}L^{\frac{c-1}{2}}(1-2^bL)}\big) \geq \varrho(0,v,t') \tag{36}$$

and inequality (21) holds. One can easily prove the uniqueness of G (see [27]). $\square$

In the next theorem, we prove the stability of the mixed type functional equations in fuzzy modular spaces for even case by using fixed point method. For a mapping $g : S \to (T, \varrho, *)$, consider

$$D_e(u,v) = g(u+2v)+g(u-2v)-4(g(u+v)-g(u-v))+24g(v)+6g(u)-3g(2v)$$

Theorem 3.2. *: Let S be a vector space, T be a real linear space, $(T, \mu, *)$ be a μ-complete b-homogeneous fuzzy modular space and $\gamma \in \{-1, 1\}$ be fixed. Assume that an even mapping $g : S \to (T, \varrho, *)$ satisfies*

$$\mu(D_e(u, v), t') \geq \varrho(u, v, t') \tag{37}$$

for all $u, v \in S$ and a given function $\varrho : S \times S \to \triangle$ in such a way that

$$\varrho(2^c u, 2^c v, 2^{4bc} L t') \geq \varrho(u, v, t') \tag{38}$$

and

$$lim_{l\to\infty} \varrho(2^{cl} u, 2^{cl} v, 2^{4bcl} t') = 1 \tag{39}$$

for all $u, v \in S$ and a constant $0 < L < \frac{1}{2^{4b}}$. Then there exists a unique quartic mapping $G : S \to (T, \mu)$ satisfying (1) and

$$\varrho\big(G(u) - g(v), \frac{t'}{2^{4b} L^{\frac{c-1}{2}} (1 - 2^b L)}\big) \geq \varrho(0, v, t'). \tag{40}$$

Proof. : Setting $u = 0$ in (37), we have

$$\mu(g(2v) - 16g(v), t') \geq \varrho(0, v, t') \tag{41}$$

so

$$\mu(\frac{g(2v)}{2^4} - g(v), t') \geq \varrho(0, v, 2^{4b} t'). \tag{42}$$

Changing v with $2^{-1} v$ in (42), we get

$$\begin{aligned} &\mu(\frac{g(2v)}{2^4} - g(2^{-1}v), \frac{t'}{2^{4b}}) \\ &\geq \varrho(0, 2^{-1}v, 2^{4b} 2^b L L^{-1} \frac{t'}{2^{4b}}) \geq \varrho(0, v, 2^{4b} L^{-1} t'). \end{aligned} \tag{43}$$

Now from (42) and (43), we get

$$\mu(\frac{g(2^c v)}{2^{4c}} - g(v), t') \geq \Phi(v, t') = \varrho(0, v, 2^{4b} L^{\frac{c-1}{2}} t'). \tag{44}$$

Assume $R = \{f : S \to (T, \mu) : f(0) = 0\}$ and define η on R as

$$\eta(f) = inf\{m > 0 : \varrho(f(v), mt') \geq \Phi(v, t')\}.$$

It is easy to prove that η is modular on L and includes the $\triangle_2-$condition with $2^b = k$ and Fatou's property. Moreover, L is $\eta-$complete(see[27]).

Let $P : R_\eta \to R_\eta$ be a function and defined as

$$P(G(v)) = \frac{G(2^c v)}{2^{4c}} \ \forall G \in R_\eta.$$

Let $j, k \in R_\eta$ and $m > 0$ be an arbitrary constant with $\eta(j - k) \leq m$. From the definition of η, we have $\mu(f(v) - k(v), mt') \geq \Phi(v, t')$ and then

$$\begin{aligned} \mu(Pf(v) - Pk(v), Lmt') &= \mu(2^{-4c} f(2^c v) - 2^{-4c} k(2^c v), Lmt') \\ &= \mu(f(2^c v) - k(2^c v), 2^{4bc} Lmt') \\ &\geq \Phi(2^c v, 2^{4bc} Lmt') \geq \Phi(v, t'). \end{aligned}$$

Hence $\eta(Pf - Pk) \leq L\eta(j - k)$, this implies that P is an $\eta-$strict contradiction. Changing v by $2^c v$ in (44), we get

$$\mu\big(\frac{g(2^{2c} v)}{2^{4c}} - g(2^c v), t'\big) \geq \Phi(2^c v, t') \tag{45}$$

and therefore

$$\begin{aligned}\mu\big(2^{-2(4c)}g(2^{2c}v)-2^{-4c}g(2^{c}v),Lt'\big) &= \big(2^{-4c}g(2^{2c}v)-g(2^{c}v),2^{4bc}Lt'\big)\\ &\geq \Phi(2^{c}v,2^{4bc}Lt')\geq\Phi(v,t'). \qquad (46)\end{aligned}$$

Now

$$\begin{aligned}&\mu\big(\frac{g(2^{2c}v)}{2^{2(4c)}}-g(v),2^{b}(Lt'+t')\big)\\ \geq\;& \mu\big(\frac{g(2^{2c}v)}{2^{2(4c)}}-\frac{g(2^{c}v)}{2^{4c}},Lt'\big)\wedge\mu\big(\frac{g(2^{c}v)}{2^{4c}}-g(v),t'\big)\\ \geq\;& \Phi(v,t'). \qquad (47)\end{aligned}$$

In (47), changing v by $2^{c}v$ and $2^{b}(Lt'+t')$ by $2^{b(1+4c)}(L^{4}t'+L^{3}t')$, we obtain

$$\begin{aligned}&\mu\big(\frac{g(2^{3c}v)}{2^{2(4c)}}-g(2^{c}v),2^{4bc}2^{b}(L^{4}t'+L^{3}t')\big)\\ \geq\;& \Phi(2^{c}v,2^{4bc}Lt')\geq\Phi(v,t'). \qquad (48)\end{aligned}$$

Therefore

$$\mu\big(\frac{g(2^{3c}v)}{2^{3(4c)}}-\frac{g(2^{c}v)}{2^{4c}}\big)(2^{c}(L^{4}t'+L^{3}t'))\geq\Phi(v,t'). \qquad (49)$$

This means

$$\begin{aligned}&\mu\big(\frac{g(2^{3c}v)}{2^{3(4c)}}-g(v),2^{b}(2^{b}(L^{4}t'+L^{3}t')+t')\big)\\ \geq\;& \mu\big(\frac{g(2^{3c}v)}{2^{3(4c)}}-\frac{g(2^{c}v)}{2^{4c}},2^{b}(L^{4}t'+L^{3}t')\big)\wedge\mu\big(\frac{g(2^{c}v)}{2^{4c}}-g(v),t'\big)\\ \geq\;& \Phi(v,t'). \qquad (50)\end{aligned}$$

Now, we generalizing the above inequality, we have

$$\mu\big(\frac{g(2^{cl}v)}{2^{4(cl)}}-g(v),(2^{b}L)^{l-1}+2^{b}\sum_{i=1}^{l-1}(2^{b}L^{i-1}t')\geq\Phi(v,t') \qquad (51)$$

where l is a positive integer. So we have

$$\begin{aligned}\eta(P^{l}g-g) &\leq (2^{b}L)^{l-1}+2^{b}\sum_{i=1}^{l-1}(2^{b}L)^{i-1}\\ &\leq 2^{b}\sum_{i=1}^{l-1}(2^{b}L)^{i-1}\\ &\leq \frac{2^{b}}{1-2^{b}L}. \qquad (52)\end{aligned}$$

Now, It is easily to prove that $\{P^{l}g\}$ is $\eta-$convergent to $G\in R_{\eta}$ (see[27]). Therefore (34) becomes

$$\eta(G-g)\leq\frac{2^{b}}{1-2^{b}L} \qquad (53)$$

this means

$$\mu(G(v)-g(v),\frac{2^{b}}{1-2^{b}L}t')\geq\Phi(v,t')=\varrho(0,v,2^{4b}L^{\frac{c-1}{2}}t') \qquad (54)$$

and so,

$$\mu\big(G(v)-g(v), \frac{t'}{2^{4b}L^{\frac{c-1}{2}}(1-2^bL)}\big) \geq \varrho(0,v,t')$$

and inequality (40) holds. One can easily prove the uniqueness of G (see [27]). $\square$

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