



A new generalized Bell wavelet and its applications for solving linear and nonlinear integral equations

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Abstract

In this study, a new fractional function based on the Bell wavelet is defined to solve the Fredholm and Volterra integral equations. The aim of this study is to approximate the unknown function of the linear (or nonlinear) integral equations by truncating the Bell wavelet series. Firstly, by using generalized Bell polynomials, the fractional Bell wavelet functions are defined. Secondly, the operational matrices are derived and transformed into matrix form. Then, a numerical scheme is developed to apply both linear and nonlinear test problems from the literature, including equations with exact solutions. By applying the generalized Bell wavelet in combination with the collocation method, the original problems are converted into a system of linear or nonlinear algebraic equations. These equations are then solved using classical techniques to determine the unknown coefficients. To evaluate the effectiveness of the proposed approach, test problems are compared with results from several established methods, and the outcomes are visually represented. This method demonstrates significantly improved accuracy compared to those found in the existing literature.

Keywords Generalized Bell polynomials · Generalized Bell wavelets · Fredholm integral equations · Collocation points · Volterra integral equations

Mathematics Subject Classification 45G15 · 45B05 · 45D05 · 65D30 · 65T60

1 Introduction

Integral equations arise in various fields, such as physics, biology, engineering, and beyond, both linear and nonlinear in nature. Over recent decades, significant research has been dedicated to their theory and applications. This attention has led to the development of numerous numerical methods tailored to solve integral equations effectively (Ahmad et al. 2021; Ullah

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et al. 2020; Almalahi et al. 2022; Mirzaee and Alipour 2019, 2018). Integral equations find relevance in diverse areas, including mathematical physics, such as diffraction problems and scattering in quantum mechanics, as well as applications in electromagnetics, electroelasticity, transport theory, mathematical epidemiology, and many others (Lechleiter and Nguyen 2013; Chew et al. 2022; Spiga et al. 1984; Lopez and Coutinho 2000). Notably, integral equations are prevalent in plasma and nuclear physics with wide-ranging implications (Melazzi and Lancellotti 2014; Neudecker et al. 2021). Among the several numerical techniques proposed for solving these equations, the collocation method stands out as a favored approach, with numerous papers exploring its efficacy, including references (Yadav et al. 2023a; Ahmed and Jahan 2024; Srivastava et al. 2020; Wang et al. 2024).

Numerical techniques for solving integral equations, particularly Fredholm and Volterra equations, are of significant interest due to their wide-ranging applications. Mirzaee et al. (Mirzaee and Hadadiyan 2016; Mirzaee 2017) discussed the numerical solution of Volterra-Fredholm integral equations by using modification of hat functions and Bell polynomials. Solhi et al. (Solhi et al. 2024) solved the stochastic fractional Volterra integro-differential equations by the enhanced moving least squares method. Integral equations are employed to represent a range of physical phenomena, including radiation, convection, and heat conduction, while also finding application in numerous scientific and engineering domains. Examples abound, ranging from biological species cohabitation to scenarios like fish propagation in new environments (Pramanik 2021), Volterra's population growth model (Srivastava et al. 2020), and radiation processes (Gao et al. 2021). A wide range of boundary and initial value problems can be rephrased in terms of Fredholm or Volterra integral equations, prompting researchers to develop diverse numerical methodologies aimed at approximating solutions for both linear and nonlinear integral equations.

In recent years, wavelet theory has gathered considerable attention from researchers across various scientific and engineering domains, both in theory and application. Notably, it boasts several appealing attributes, such as accurate representation of polynomials up to a certain degree, compact support, orthogonality, and the ability to represent diverse functions across different levels of resolution. These functions offer the advantage of being versatile in applications such as time-frequency analysis, waveform representation and segmentation, signal processing, optimal control, image processing, edge detection, numerical analysis, system modeling, and the development of fast algorithms for efficient implementation. Researchers have explored a range of techniques for numerical approximations by utilizing wavelets, including the wavelet-based finite element method (Díaz et al. 2009), the wavelet collocation method (Ahmed et al. 2024; Amin et al. 2020), the wavelet meshless method (Liu and Cen 2008), and the wavelet Galerkin method (Jang et al. 2004). Early investigations into these methods can be found in (Yadav et al. 2023b). Additionally, wavelets have been employed for tasks such as numerical integration (Yadav et al. 2023a; Ahmed and Jahan 2024; Yasmeen et al. 2023) and solving integral equations numerically.

Recently, a study by the author Kazem et al. (2013) introduced a transformation, $t = t^\alpha$ ($\alpha > 0$), applied to shifted Legendre polynomials, aiming to enhance the results in solving fractional order differential equations. Similarly, in another work Bhrawy et al. 2014, the same transformation was utilized with generalized Laguerre polynomials to approximate fractional differential equations. Additionally, other studies Yadav et al. (2024), Yüzbaşı (2024), Rahimkhani et al. (2016), Mohammadi and Cattani (2018), Izadi et al. (2022) and Izadi et al. (2024) employed Fibonacci wavelets, Bell functions, Bernoulli wavelets, Legendre functions, Morgan-Voyce polynomials and Hermite functions to derive numerical solutions for fractional differential or integral equations. Izadi and Srivastava 2021 applied generalized Bessel quasilinearization method for solving Bratu and Lane–Emden-Type equations of

arbitrary order. This motivates us to further expand and generalize the recently introduced functions defined on Bell wavelets, with the aiming of tackling both linear and nonlinear integral equations. The Bell wavelet method possesses key properties and characteristics that make it highly effective for solving integral and differential equations. Unlike Bell polynomials, which lead to a full-rank coefficient matrix, Bell wavelets exhibit the properties of locality and sparsity. The advantage of employing generalized Bell wavelets significantly lowers the computational expenses of wavelet approximations. These characteristics enable the Bell wavelet method to provide improved approximate solutions. The method stands out for its precision, efficiency, and ease of implementation in solving such equations. Additionally, this method yields a family of wavelet representations parameterized by α , which demonstrates improved accuracy for the numerical solutions of integral equations.

In this study, the generalized Bell wavelet is defined for solving the linear (or nonlinear) Fredholm and Volterra integral equations. Operational matrices of integration are constructed using this generalized Bell function basis. The given problems are then transformed into a set of algebraic equations to find the unknown Bell wavelet coefficients in order to approximate the integral equation. The approximate solution's accuracy and efficiency are also tested for different values of the parameter α . The proposed method is far better than the existing methods in the literature and is computationally based, which makes it easy to implement in solving linear and nonlinear integral equations.

The paper is organized as follows: Sect. 2 provides the essential preliminary knowledge of generalized Bell polynomials and wavelets and their function approximations. In Sects. 3 and 4, we discussed the description of methods for both linear (and nonlinear) Fredholm integral equations (FIEs) and Volterra integral equations (VIEs) and the convergence analysis. Section 5 is dedicated to presenting our numerical simulations and discussion, showcasing the efficacy of the presented methodology through illustrative examples. Lastly, Sect. 6 gives a brief conclusion and future directions.

2 Generalized Bell wavelet

Generalized Bell polynomials

According to Yüzbaşı (2022), the Bell polynomials $\mathcal{B}_n^\alpha(z)$ of order n , are defined by

$$\mathcal{B}_n^\alpha(z) = \sum_{k=0}^n S(n, k) z^{k\alpha},$$

here $0 < \alpha < 1$, $n = 0, 1, 2, \dots, N$ (N is any chosen $\mathbb{Z}^+$), $S(n, k)$ represents the second kind Stirling number (Srivastava 2000) and define as:

$$S(n, k) = \sum_{i=0}^k \frac{(-1)^i}{k!} \binom{k}{i} (k-i)^n.$$

Using the Bell polynomials properties, consider the vector $\mathbf{B}_{N,\alpha}(z) = [\mathcal{B}_0^\alpha(z), \dots, \mathcal{B}_N^\alpha(z)]^T$.

Lemma 2.1 (Yüzbaşı 2022) *The vector $\mathbf{B}_{N,\alpha}(z)$ can be expressed as*

$$\mathbf{B}_{N,\alpha}(z) = \mathbf{S}_N \mathbf{Y}_{N,\alpha}(z),$$

where

$$\mathbf{Y}_{N,\alpha}(z) = [1, z^\alpha, \dots, z^{N\alpha}]^T,$$

$$\mathbf{S}_N = \begin{bmatrix} S(0, 0) & 0 & \cdots & 0 \\ S(1, 0) & S(1, 1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S(n, 0) & S(n, 1) & \cdots & S(n, n) \end{bmatrix}.$$

The first five generalized Bell polynomials are

$$\begin{aligned} \mathcal{B}_0^\alpha(z) &= 1, \\ \mathcal{B}_1^\alpha(z) &= z^\alpha, \\ \mathcal{B}_2^\alpha(z) &= z^{2\alpha} + z^\alpha, \\ \mathcal{B}_3^\alpha(z) &= z^{3\alpha} + 3z^{2\alpha} + z^\alpha, \\ \mathcal{B}_4^\alpha(z) &= z^{4\alpha} + 6z^{3\alpha} + 7z^{2\alpha} + z^\alpha. \end{aligned}$$

The Bell wavelets $\psi_{mn}(\eta) = \psi(k, m, n, \eta)$ have four arguments: $k \in \mathbb{Z}^+$, $m = 1, \dots, 2^{k-1}$, the η is the normalized time and degree of the Bell polynomials is given by $n = 0, 1, \dots, M-1$. On the interval $[0, 1)$, Bell wavelets are given by

$$\psi_{mn}(\eta) = \begin{cases} 2^{\frac{k-1}{2}} \mathcal{B}_n(2^{k-1}\eta - m + 1), & \frac{m-1}{2^{k-1}} \leq \eta < \frac{m}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

Generalized Bell wavelet

Here, we construct the the generalized Bell wavelet by changing the variable $\eta = z^\alpha$, ($\alpha > 0$), on the Bell wavelet. So the generalized Bell wavelet $\psi_{mn}(z^\alpha)$ is denoted by $\psi_{mn}^\alpha(z)$. The following relation is obtained

$$\psi_{mn}^\alpha(z) = \begin{cases} 2^{\frac{k-1}{2}} \mathcal{B}_n(2^{k-1}z^\alpha - m + 1), & \frac{m-1}{2^{k-1}} \leq z^\alpha < \frac{m}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

The generalized Bell wavelets basis for $k = 2$, $M = 3$ are

$$\begin{aligned} \psi_{1,0}^\alpha(z) &= \begin{cases} \sqrt{2}, & 0 \leq z^\alpha < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \\ \psi_{1,1}^\alpha(z) &= \begin{cases} 2\sqrt{2}z^\alpha, & 0 \leq z^\alpha < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \\ \psi_{1,2}^\alpha(z) &= \begin{cases} 4\sqrt{2}z^{2\alpha} + 2\sqrt{2}z^\alpha, & 0 \leq z^\alpha < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \\ \psi_{2,0}^\alpha(z) &= \begin{cases} \sqrt{2}, & \frac{1}{2} \leq z^\alpha < 1, \\ 0, & \text{otherwise.} \end{cases} \\ \psi_{2,1}^\alpha(z) &= \begin{cases} 2\sqrt{2}z^\alpha - \sqrt{2}, & \frac{1}{2} \leq z^\alpha < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$$\psi_{2,2}^\alpha(z) = \begin{cases} 4\sqrt{2}z^{2\alpha} - 2\sqrt{2}z^\alpha, & \frac{1}{2} \leq z^\alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix will be for $k = 2, M = 3, \alpha = 0.7$

$$\psi_{6 \times 6}^{0.7} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 0.4967 & 1.0717 & 1.5325 & 0 & 0 & 0 \\ 0.6712 & 1.8840 & 3.1931 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0.5253 & 0.8983 & 1.2471 \\ 0 & 0 & 0 & 0.7204 & 1.4689 & 2.3468 \end{bmatrix}.$$

Function approximation

A function $\mu(z) \in L^2[0, 1]$ could be represented in form of generalized Bell wavelets as:

$$\mu(z) = \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}^+} u_{mn} \psi_{mn}^\alpha(z).$$

By truncating above equation, we get

$$\mu(z) \simeq \sum_{m=1}^{2^{k-1}} \sum_{n=0}^{M-1} u_{mn} \psi_{mn}^\alpha(z), \quad (2.3)$$

where

$$u_{m,n} = \int_0^1 \mu(z) \psi_{m,n}(z) z^{\alpha-1} dz,$$

are the coefficients of generalized Bell wavelet. The Eq. (2.3) can be expressed as a matrix:

$$\mu(z) = U^T \Psi(z), \quad (2.4)$$

here we define the row vector U as:

$$U = [u_{1,0}, \dots, u_{1,M-2}, u_{1,M-1}, u_{2,0}, \dots, u_{2,M-2}, u_{2,M-1}, \dots, u_{2^{k-1},0}, u_{2^{k-1},1}, \dots, u_{2^{k-1},M-1}]^T. \quad (2.5)$$

The matrix $\Psi(z)$ in (2.4) is $1 \times 2^{k-1}M$ ($\hat{m}=2^{k-1}M$) order Bell wavelet matrix is as

$$\Psi^\alpha(z) = [\psi_{1,0}^\alpha, \dots, \psi_{1,M-2}^\alpha, \psi_{1,M-1}^\alpha, \psi_{2,0}^\alpha, \dots, \psi_{2,M-2}^\alpha, \psi_{2,M-1}^\alpha, \dots, \psi_{2^{k-1},0}^\alpha, \psi_{2^{k-1},1}^\alpha, \dots, \psi_{2^{k-1},M-1}^\alpha]^T. \quad (2.6)$$

Finally, taking the collocation points:

$$z_\ell = \frac{2^\ell - 1}{2^k M}, \quad \ell = 1, 2, \dots, 2^{k-1}M. \quad (2.7)$$

The integrals that will be used to solve VIEs are as follows:

$$q_{m,n}^\alpha(z) = \int_0^z \psi_{mn}^\alpha(t) dt, \quad (2.8)$$

and the elements of the matrices for (2.8) are given by

$$(q_{m,n}^\alpha)_{2^{k-1} \times M} = \begin{pmatrix} q_{1,0}^\alpha(z) & q_{1,1}^\alpha(z) & \cdots & q_{1,M-1}^\alpha(z) \\ q_{2,0}^\alpha(z) & q_{2,1}^\alpha(z) & \cdots & q_{2,M-1}^\alpha(z) \\ \vdots & \vdots & \ddots & \vdots \\ q_{2^{k-1},0}^\alpha(z) & q_{2^{k-1},1}^\alpha(z) & \cdots & q_{2^{k-1},M-1}^\alpha(z) \end{pmatrix}_{2^{k-1} \times M}.$$

Remark 2.1 For $2^{k-1} \times M$ -vector H , we have

$$\psi^\alpha(z)(\psi^\alpha(z))^T H = \tilde{H} \psi^\alpha(z),$$

here, the generalized Bell wavelet coefficient matrix is denoted by $\psi^\alpha(z)$ and matrix $\tilde{H}$ is of $2^{k-1}M \times 2^{k-1}M$ order shown below

$$\tilde{H} = \psi^\alpha(z) \bar{H} (\psi^\alpha(z))^{-1},$$

here $\bar{H} = \text{diag}((\psi^\alpha(z))^{-1} H)$. Also, for a $2^{k-1}M \times 2^{k-1}M$ order Q matrix, we consider

$$(\psi^\alpha(z))^T Q \psi^\alpha(z) = \hat{Q}^T \psi^\alpha(z),$$

where, $\hat{Q}^T = V(\psi^\alpha(z))^{-1}$ and V is a $2^{k-1}M$ vector as $V = \text{diag}((\psi^\alpha(z))^T Q \psi^\alpha(z))$.

Remark 2.2 Consider u as an analytic function on $\mathbb{R}$. If we expand $\lambda(z)$ using generalized Bell wavelet, the expansion is represented by $X^T \psi^\alpha(z)$, then

$$u(\lambda(z)) \simeq u(X^T) \psi^\alpha(z), \quad (2.9)$$

where, $u(X^T) = [u(X_1), u(X_2), \dots, u(X_{2^{k-1}M})]$.

Remark 2.3 Consider u as an analytic function on $\mathbb{R}$. If we expand $\lambda(z)$ using generalized Bell wavelet, the expansion is represented by $X^T \psi^\alpha(z)$, then

$$u(\lambda(z)) \simeq u(\tilde{X}^T)(\psi^\alpha)^{-1}(z) \psi^\alpha(z), \quad (2.10)$$

where, $\tilde{X}^T = X^T \psi^\alpha(z)$, $\psi^\alpha(z)$ is the generalized Bell wavelet coefficient matrix given in (2.6) and $u(X^T)$ is given in Remark 2.2.

3 Description of generalized Bell wavelet method

In this section, we aim to present a methodology for obtaining an approximate solution to the FIEs and VIEs using generalized Bell wavelets.

3.1 FIEs

Case 1: Linear integral equation

Consider the linear FIEs as follows:

$$\mu(z) = \lambda(z) + \int_0^1 k(z, t) \mu(t) dt. \quad (3.1)$$

Now, approximating $\mu(z)$, $\lambda(z)$, $k(z, t)$ w.r.t generalized Bell wavelet as:

$$\mu(z) \simeq U^T \psi^\alpha(z) = U(\psi^\alpha)^T(z), \quad (3.2)$$

the U vector is the undefined Bell wavelet coefficient.

$$\lambda(z) \simeq \Lambda^T \psi^\alpha(z) = \Lambda(\psi^\alpha)^T(z), \quad (3.3)$$

and

$$k(z, t) \simeq (\psi^\alpha)^T(z) \mathcal{K} \psi^\alpha(t) = (\psi^\alpha)^T(t) \mathcal{K}^T \psi^\alpha(z), \quad (3.4)$$

where vector Λ is the generalized Bell wavelet coefficient and $\mathcal{K}$ is the generalized Bell wavelet matrices. Now, put equations (3.2), (3.3), and (3.4) in (3.1), we have

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + U^T \left(\int_0^1 \psi^\alpha(t) (\psi^\alpha)^T(t) dt \right) \mathcal{K} \psi^\alpha(z).$$

Using, $\int_0^1 \psi(t) \psi^T(t) dt = 1$, the following equation obtain

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + U^T \mathcal{K} \psi^\alpha(z).$$

Therefore,

$$U^T - U^T \mathcal{K} = \Lambda^T. \quad (3.5)$$

So the undefined generalized Bell wavelet coefficient U is obtained by solving the above (3.5) system of equations. And simultaneously, the approximate solution of Eq. (3.1) is obtained by substituting the U in equation (3.2).

Case 2: Nonlinear integral equation

Let us consider the nonlinear FIE as follows:

$$\mu(z) = \lambda(z) + \int_0^1 k(z, t) [\mu(t)]^l dt, \quad l > 0. \quad (3.6)$$

Now, approximating $\mu(z)$, $\lambda(z)$, $k(z, t)$ and $[\mu(z)]^l$ w.r.t generalized Bell wavelet as:

$$\mu(z) \simeq U^T \psi^\alpha(z) = U(\psi^\alpha)^T(z), \quad (3.7)$$

the U vector is the undefined Bell wavelet coefficient.

$$\lambda(z) \simeq \Lambda^T \psi^\alpha(z) = \Lambda(\psi^\alpha)^T(z) \quad (3.8)$$

and

$$k(z, t) \simeq (\psi^\alpha)^T(z) \mathcal{K} \psi^\alpha(t) = (\psi^\alpha)^T(t) \mathcal{K}^T \psi^\alpha(z) \quad (3.9)$$

and

$$[\mu(z)]^l \simeq (U^*)^T \psi^\alpha(z) = (U^*)(\psi^\alpha)^T(z), \quad (3.10)$$

where vector Λ is the generalized Bell wavelet coefficient, $\mathcal{K}$ is the generalized Bell wavelet matrices and column vector U^* function is the elements of U . By putting equations (3.7), (3.8), (3.9) and (3.10) in (3.6), we obtain

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + (U^*)^T \left(\int_0^1 \psi^\alpha(t) (\psi^\alpha)^T(t) dt \right) \mathcal{K} \psi^\alpha(z).$$

Using, $\int_0^1 \psi(t) \psi^T(t) dt = 1$, the following equation obtain

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + (U^*)^T \mathcal{K} \psi^\alpha(z).$$

Therefore,

$$U^T - (U^*)^T \mathcal{K} = \Lambda^T. \quad (3.11)$$

So the undefined generalized Bell wavelet coefficient U is obtained by solving the above (3.11) system of nonlinear equation.

3.2 VIEs

Case 1: Linear integral equation

Consider the linear VIE

$$\mu(z) = \lambda(z) + \int_a^z k_1(z, t) \mu(t) dt. \quad (3.12)$$

Without loss of generality, let us take $a = 0$. Approximating $\lambda(z)$, $\mu(z)$ and $k_1(z, t)$, w.r.t generalized Bell wavelets as:

$$\mu(z) \simeq U^T \psi^\alpha(z) = U(\psi^\alpha)^T(z), \quad (3.13)$$

where U is defined in (2.5) and is the undefined vector to be obtain,

$$\lambda(z) \simeq \Lambda^T \psi^\alpha(z) = \Lambda(\psi^\alpha)^T(z), \quad (3.14)$$

and

$$k_1(z, t) \simeq (\psi^\alpha)^T(z) \mathcal{K}_1 \psi^\alpha(t) = (\psi^\alpha)^T(t) \mathcal{K}_1^T \psi^\alpha(z), \quad (3.15)$$

where U and Λ vectors are generalized Bell wavelet coefficient and $\mathcal{K}_1$ is the generalized Bell wavelet matrix. Now, by putting (3.13), (3.14) and (3.15) in (3.12), we get

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1^T \left(\int_0^z \psi^\alpha(t) (\psi^\alpha)^T(t) U dt \right).$$

From Remark 2.1, we have

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1^T \left(\int_0^z \tilde{U} \psi^\alpha(z) dt \right),$$

where $\tilde{U}$ matrix is of $2^{k-1}M \times 2^{k-1}M$ order. Using the operational matrix of generalized Bell wavelet, we have

$$U^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1^T \tilde{U} q^\alpha \psi^\alpha(z).$$

Let $Z_1 = \mathcal{K}_1^T \tilde{U} q^\alpha$. Again from Remark 2.1, we have

$$U^T \psi^\alpha(z) - \hat{Z}_1^T \psi^\alpha(z) = \Lambda^T \psi^\alpha(z),$$

where $\hat{Z}_1, \tilde{U}$ are linear functions of U of $2^{k-1}M \times 2^{k-1}M$ order and it is $\forall t \in [0, 1)$, so

$$U^T - \hat{Z}_1^T = \Lambda^T. \quad (3.16)$$

By solving the above (3.16) system, the undefined vector U is obtained. By putting the obtained U vector in Eq. (3.13), the required solution of VIE (3.12) is obtained.

Case 2: Nonlinear integral equation

Consider the nonlinear VIE

$$\mu(z) = \lambda(z) + \int_0^z k(z, t)u(\mu(t))dt, \quad (3.17)$$

as $t, z \in [0, 1)$, $u(\mu(z))$, and $k(z, t)$ is function of z and t are the known functions, and $\mu(z)$ is the undefined function. Approximating $\lambda(z)$, $\mu(z)$, and $k(z, t)$ w.r.t generalized Bell wavelet, we have

$$\mu(z) \simeq U^T \psi^\alpha(z) = U(\psi^\alpha)^T(z), \quad (3.18)$$

where U is mentioned in (2.5) which is the unknown to be determined.

$$\lambda(z) \simeq \Lambda^T \psi^\alpha(z) = \Lambda(\psi^\alpha)^T(z) \quad (3.19)$$

and

$$k(z, t) \simeq \Lambda^T \psi^\alpha(z) = (\psi^\alpha)^T(t) \mathcal{K}_1 \psi^\alpha(z), \quad (3.20)$$

where U and Λ are the coefficient of generalized Bell wavelet, and $\mathcal{K}_1$ is the wavelet matrix. By putting Eqs. (3.18), (3.19), and (3.20) into (3.17), we have

$$U^T \psi^\alpha(z) \simeq \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1 \left(\int_0^z \psi^\alpha(t) u(U^T \psi^\alpha(t)) dt \right). \quad (3.21)$$

From Remark 2.3, Eq. (3.21) can also be written as,

$$U^T \psi^\alpha(z) \simeq \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1 \left(\int_0^z \psi^\alpha(t) (\psi^\alpha)^T(t) Z dt \right), \quad (3.22)$$

where, $Z^T = u(\tilde{U}^T)(\psi^\alpha(z))^{-1}$, and $\tilde{U}^T = U^T \psi^\alpha(z)$. From (3.22) and Remark 2.1, we have

$$U^T \psi^\alpha(z) \simeq \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1 \left(\int_0^z \tilde{Z} \psi^\alpha(t) dt \right), \quad (3.23)$$

where, $\tilde{Z}$ is a matrix described in Remark 2.1. Utilizing the operational matrix of integration of generalized Bell wavelet, Eq. (3.23) becomes:

$$U^T \psi^\alpha(z) \simeq \Lambda^T \psi^\alpha(z) + (\psi^\alpha)^T(z) \mathcal{K}_1 \tilde{Z} q^\alpha \psi^\alpha(z). \quad (3.24)$$

Suppose that $\delta = \mathcal{K}_1 \tilde{Z} q^\alpha$. Again from Remark 2.1, Eq. (3.24) becomes:

$$U^T \psi^\alpha(z) - \hat{\delta} \psi^\alpha(z) \simeq \Lambda^T \psi^\alpha(z). \quad (3.25)$$

In this context, $\hat{\delta}$ represents a $2^{k-1}M$ -vector comprising a nonlinear combination of elements from U . Equation (3.25) is valid for all z within the interval $[0, 1)$. Consequently, by substituting “ $\simeq$ ” with “ $=$ ” in Eq. (3.25), it transforms into a nonlinear system as $U^T - \hat{\delta} = \Lambda^T$. The unknown vector U is obtained from solving this nonlinear system. Then, with this determined vector substituted into (3.18), we derive the approximate solution for (3.17).

4 Convergence analysis

In this section, we discuss the theoretical analysis of convergence of our approach.

Theorem 4.1 Let H be a Hilbert space and W a closed subspace of H such that $\dim W < \infty$. Consider $w_1, w_2, \dots, w_r$ as any basis for W . For any arbitrary element x in H , denote w^* as the best unique approximation to x within W . Therefore (Kreyszig 1991)

$$\|x - w^*\|_2^2 = \frac{G(x, w_1, w_2, \dots, w_n)}{G(w_1, w_2, \dots, w_n)},$$

where

$$G(x, w_1, w_2, \dots, w_n) = \begin{vmatrix} \langle x, x \rangle & \langle x, w_1 \rangle & \cdots & \langle x, w_n \rangle \\ \langle w_1, x \rangle & \langle w_1, w_1 \rangle & \cdots & \langle w_1, w_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_n, x \rangle & \langle w_n, w_1 \rangle & \cdots & \langle w_n, w_n \rangle \end{vmatrix}.$$

Before stating the theorem, we first introduce a definition that will be used within it.

Definition 4.1 (Generalized Taylor's formula) (Odibat and Shawagfeh 2007). Suppose that $\mathcal{D}^{i\alpha}\mu \in C(0, 1]$ for $i = 0, 1, \dots, m$. Then we have

$$\mu(z) = \sum_{i=0}^{m-1} \frac{z^{i\alpha}}{\Gamma(i\alpha + 1)} \mathcal{D}^{i\alpha}\mu(0^+) + \frac{z^{m\alpha}}{\Gamma(m\alpha + 1)} \mathcal{D}^{m\alpha}\mu(\xi), \quad (4.1)$$

with $0 < \xi \leq z, \forall z \in (0, 1]$. Also,

$$\left| \mu(z) - \sum_{i=0}^{m-1} \frac{z^{i\alpha}}{\Gamma(i\alpha + 1)} \mathcal{D}^{i\alpha}\mu(0^+) \right| \leq M_\alpha \frac{z^{m\alpha}}{\Gamma(m\alpha + 1)},$$

where $M_\alpha \geq \sup_{\xi \in (0, 1]} |\mathcal{D}^{m\alpha}\mu(\xi)|$.

In case of $\alpha = 1$, the generalized Taylor's formula (4.1) reduces to the classical Taylor's formula.

Theorem 4.2 Suppose for $k = 0, 1, \dots, M-1$ $\mathcal{D}^{k\alpha}\mu \in C(0, 1]$ and $\mathcal{Q}_n^\alpha = \text{span}\{\mathcal{B}_0^\alpha, \mathcal{B}_1^\alpha, \dots, \mathcal{B}_{(M-1)}^\alpha\}$. If $\mu_{\hat{n}} = \sum_{i=0}^{M-1} y_i \mathcal{B}_i^\alpha = \mathbf{Y}^T \mathcal{B}^\alpha$ is the best approximation of μ out of $\mathcal{Q}_n^\alpha$ on $\left[\frac{m-1}{2^{k-1}}, \frac{m}{2^{k-1}}\right]$, therefore

$$\|\mu - \mu_{\hat{n}}\|_2 \leq \sup_{z \in [0, 1]} \left| \mathcal{D}^{M\alpha}\mu(0^+) \right| \frac{1}{\Gamma(M\alpha + 1)} \sqrt{\frac{1}{(2M+1)\alpha}}.$$

Proof Suppose Generalized Taylor's formula, then

$$\hat{y}(z) = \sum_{i=0}^{M-1} \frac{z^{i\alpha}}{\Gamma(i\alpha + 1)} \mathcal{D}_{i\alpha}\mu(0^+).$$

Next,

$$|\mu(z) - \hat{y}(z)| \leq \frac{z^{M\alpha}}{\Gamma(M\alpha + 1)} \sup_{z \in I_{k,m}} |\mathcal{D}^{M\alpha}\mu(0^+)|,$$

here $I_{k,m} = \left[\frac{m-1}{2^{k-1}}, \frac{m}{2^{k-1}}\right]$. Then,

$$\|\mu - \mu_{\hat{n}}\|_2^2 = \int_0^1 w(z) [\mu(z) - U^T \Psi^\alpha(z)]^2 dz$$

$$\begin{aligned}
&= \sum_{m=1}^{2^{k-1}} \int_{\frac{m-1}{2^{k-1}}}^{\frac{m}{2^{k-1}}} w_m(z) [\mu(z) - U^T \Psi^\alpha(z)]^2 dz \\
&= \sum_{m=1}^{2^{k-1}} \int_{\frac{m-1}{2^{k-1}}}^{\frac{m}{2^{k-1}}} w_m(z) [\mu(z) - \mathbf{Y}^T \mathcal{B}^\alpha(z)]^2 dz,
\end{aligned}$$

where $\mathbf{Y}^T \mathcal{B}^\alpha$ represents any function of $(M-1)\alpha$ order that approximates μ on $I_{k,m}$ with the interpolation error bound as:

$$|\mu(z) - \mathbf{Y}^T \mathcal{B}^\alpha(z)| \leq \frac{z^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{z \in I_{k,m}} |\mathcal{D}^{M\alpha} \mu(0^+)|.$$

Thus, based on the aforementioned relations, we derive:

$$\begin{aligned}
\|\mu - \mu_{\hat{n}}\|_2^2 &= \sum_{m=1}^{2^{k-1}} \int_{\frac{m-1}{2^{k-1}}}^{\frac{m}{2^{k-1}}} w_m(z) \left(\frac{z^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{z \in I_{k,m}} |\mathcal{D}^{M\alpha} \mu(0^+)| \right)^2 dz \\
&= \int_0^1 w(z) \left(\frac{z^{M\alpha}}{\Gamma(M\alpha+1)} \sup_{z \in I_{k,m}} |\mathcal{D}^{M\alpha} \mu(0^+)| \right)^2 dz \\
&= \frac{1}{\Gamma(M\alpha+1)^2 (2M+1)\alpha} \left(\sup_{z \in [0,1]} |\mathcal{D}^{M\alpha} \mu(0^+)| \right)^2.
\end{aligned}$$

□

Theorem 4.3 Consider $\mu \in L^2[0, 1]$ is approximated by μ_m as (Keshavarz et al. 2016)

$$\mu(z) \simeq \mu_m(z) = \sum_{i=0}^{m-1} u_i \mathcal{B}_i^\alpha(z) = U^T \mathcal{B}^\alpha(z),$$

where

$$\mathcal{B}^\alpha(z) = \begin{bmatrix} \mathcal{B}_0^\alpha(z) \\ \mathcal{B}_1^\alpha(z) \\ \vdots \\ \mathcal{B}_{m-1}^\alpha(z) \end{bmatrix}, \quad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}.$$

Consider

$$J_m(\mu) = \int_0^1 [\mu(z) - \mu_m(z)]^2 dz,$$

so, we get

$$\lim_{m \rightarrow \infty} J_m(\mu) = 0.$$

5 Examples and discussions

In this section, some numerical examples are given to demonstrate the applicability and accuracy of our method. All calculations were performed through utilizing MATLAB software Version R2024a. For error estimation, the comparison between the approximate and exact

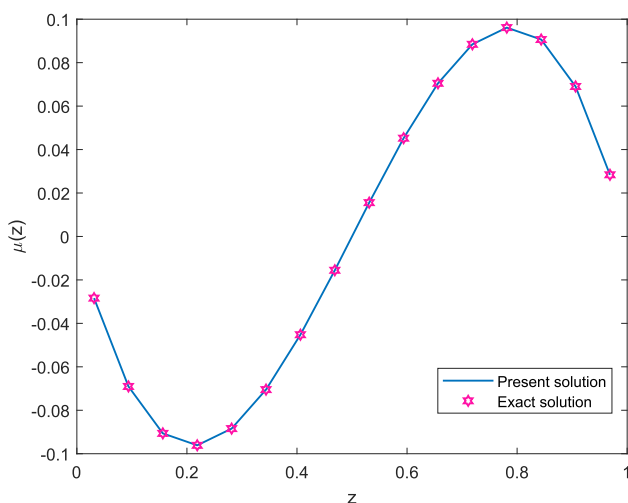


Fig. 1 Comparison of generalized Bell wavelet solution and exact solution for Problem 5.1

solutions of FEIs and VIEs is carried out by evaluating the absolute error at selected points. The absolute error is defined as:

$$e(z) = |\mu(z) - \mu^*(z)|, \quad z \in [0, 1),$$

where $\mu(z)$ represents the exact solution and $\mu^*(z)$ denotes the approximate solution.

Test Problem 5.1 First, consider the linear FIE:

$$\mu(z) = -2z^3 + 3z^2 - z + \int_0^1 (z^2 - z - t^2 + t)\mu(t)dt. \quad (5.1)$$

The exact solution for (5.1) is $\mu(z) = -2z^3 + 3z^2 - z$. Figure 1 presents the comparison curves between the present generalized Bell wavelet method solution and the exact solution for $M = 4, k = 2$ at $\alpha=1$. The absolute error results at $M = 4, k = 2$, and $\alpha=1$ between the exact and present methods are given by Fig. 2. A comprehensive analysis of the impact of the parameter α at various points is presented in Table 1. It is noteworthy that certain values of α yield significantly better results compared to integer values of α . Additionally, Tables 2 and 3 provide a comparative overview of our method against the Fibonacci method (Yadav et al. 2023a) and the Hermite method (Maleknejad and Yousefi 2006) across different resolutions. Notably, an increase in collocation points corresponds to a decrease in absolute error. The results clearly indicate that the generalized Bell wavelet method outperforms existing techniques, offering more effective and precise solutions. The process takes 2.812 s of CPU time.

Test Problem 5.2 Next, consider the linear FIE

$$\mu(z) = \sin(2\pi z) + \int_0^1 \cos(z)\mu(t)dt. \quad (5.2)$$

The exact solution of (5.2) is $\mu(z) = \sin(2\pi z)$. Figures 3 and 4 present the comparison of the present method with the exact solution and the absolute error plot for $M = 4, k = 2$ at $\alpha=1$, respectively. At $M = 3, k = 2$, and $M = 4, k = 2$ Table 4 shows the effectiveness

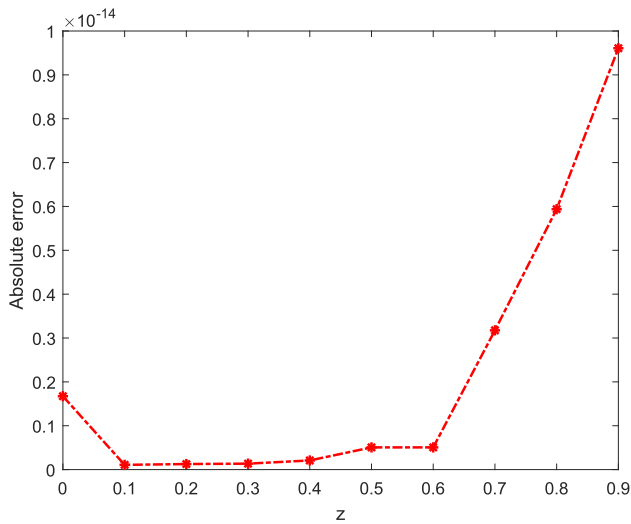


Fig. 2 At $M = 4$, $k = 2$ absolute error function plot for Problem 5.1

Table 1 Absolute error at $\alpha = 0.25$, 0.5 and 0.85 of Problem 5.1

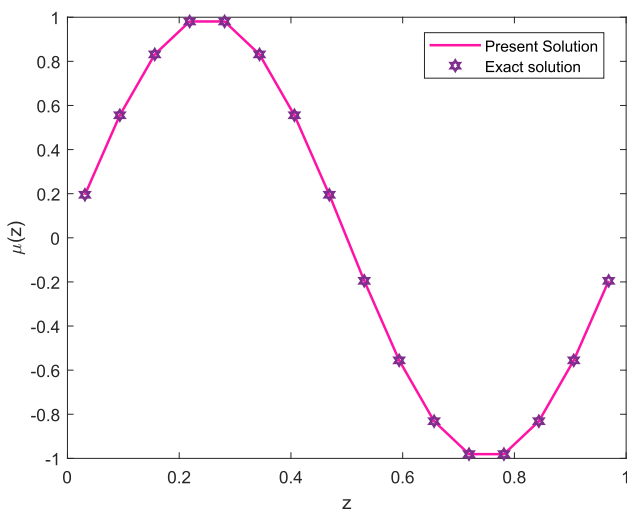
z	$M = 3, k = 2$			$M = 4, k = 2$		
	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.85$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.85$
0.1	1.0183e-15	4.5103e-17	4.5536e-18	6.1448e-15	5.318e-16	4.9223e-17
0.2	1.4433e-15	1.0647e-16	1.3531e-16	5.9119e-15	1.9741e-15	8.5912e-16
0.3	1.5735e-15	1.6658e-16	2.2107e-16	5.5216e-15	3.5714e-15	1.9661e-15
0.4	1.7828e-15	1.9136e-16	2.2107e-16	4.4253e-15	5.2467e-15	3.5388e-15
0.5	1.9483e-16	6.0173e-17	1.4713e-16	1.9054e-14	1.3434e-14	1.5337e-15
0.6	8.6042e-16	1.7347e-18	2.8753e-16	2.6909e-14	3.2863e-14	1.6508e-15
0.7	7.4523e-15	4.7735e-16	3.5475e-16	3.0267e-14	4.8905e-14	1.2817e-15
0.8	1.1883e-15	3.0184e-16	3.5475e-16	3.4042e-14	6.9e-14	7.8041e-16

Table 2 Absolute error comparisons with Fibonacci method (Yadav et al. 2023a) of Problem 5.1

z	$M = 3, k = 2$		$M = 4, k = 2$	
	Present method	Fibonacci method (Yadav et al. 2023a)	Present method	Fibonacci method (Yadav et al. 2023a)
0.125	3.361e-17	2.4869e-14	1.0072e-16	1.1369e-13
0.25	3.3719e-16	5.6843e-14	1.2533e-16	1.7053e-13
0.375	2.6433e-15	2.0073e-13	1.3704e-16	2.2737e-13
0.5	7.1054e-15	1.6343e-13	2.3939e-16	1.1368e-13
0.625	2.6411e-15	7.2830e-14	7.7342e-16	0
0.75	2.6455e-15	5.6843e-14	2.8291e-15	0
0.875	1.3211e-15	1.1013e-13	6.2838e-15	0

Table 3 Max. absolute error comparison of Problem 5.1

$\hat{m}$	Present method	Fibonacci method (Yadav et al. 2023a)	Hermite method (Maleknejad and Yousefi 2006)
4	3.7688e-14	0	1.3363e-10
8	1.1032e-14	2.2737e-13	3.7916e-10

**Fig. 3** Comparison of generalized Bell wavelet method with exact solution for Problem 5.2

and accuracy of the generalized Bell wavelet approach compared to the Fibonacci method (Yadav et al. 2023a) in terms of absolute error. Table 5 shows the comparison of max. absolute error with the Fibonacci method (Yadav et al. 2023a) and Hermite method (Maleknejad and Yousefi 2006). To analyze the effect of α in non-integer order on the approximate solutions, Table 6 is presented. The data presented in the tables and figures clearly demonstrates that the current method significantly outperforms both the Fibonacci method (Yadav et al. 2023a) and the Hermite method (Maleknejad and Yousefi 2006).

Test Problem 5.3 Consider the nonlinear FIE

$$\mu(z) = \sin(\pi z) + \frac{1}{5} \int_0^1 \cos(\pi z) \sin(\pi t) (\mu(t))^3 dt. \quad (5.3)$$

The exact solution of (5.3) is $\mu(z) = \sin(\pi z) + \frac{20-\sqrt{391}}{3} \cos(\pi z)$. The present approach outcomes are compared with exact solutions as shown by Fig. 5. Table 7 shows the error comparison of the generalized Bell wavelet method with the Triangular factorization method (Maleknejad et al. 2010) at different resolutions. Table 7 clearly demonstrates that the current method significantly surpasses the Triangular factorization method (Maleknejad et al. 2010) in terms of accuracy. Table 8 presents the error comparisons at $\alpha = 0.5, 0.75$, and 0.95 for $M = 2, k = 2$; $M = 3, k = 2$ and $M = 4, k = 3$. The process takes 2.755 s of CPU time.

Table 4 Absolute error comparisons with Fibonacci method (Yadav et al. 2023a) of Problem 5.2

z	$M = 3, k = 2$		$M = 4, k = 2$	
	Present method	Fibonacci method (Yadav et al. 2023a)	Present method	Fibonacci method (Yadav et al. 2023a)
0.125	3.4562e-13	2.9559e-12	3.2464e-13	1.8758e-12
0.25	3.0506e-12	3.2969e-12	3.1548e-13	1.9326e-12
0.375	5.5161e-12	3.9790e-12	2.9754e-13	2.2453e-12
0.5	7.7198e-12	3.4106e-12	2.7159e-13	2.1885e-12
0.625	2.5105e-12	2.5295e-12	2.3818e-13	1.4210e-12
0.75	5.5298e-12	2.2737e-12	2.3818e-13	1.9326e-12
0.875	9.543e-12	1.5064e-12	1.9806e-13	2.7853e-12

Table 5 Max. absolute error comparison with Fibonacci method (Yadav et al. 2023a) and Hermite method (Maleknejad and Yousefi 2006) of Problem 5.2

$\hat{m}$	Present method	Fibonacci method (Yadav et al. 2023a)	Hermite method (Maleknejad and Yousefi 2006)
4	3.97781e-13	4.9249e-13	2.8415e-02
8	3.2464e-13	3.0892e-12	2.3888e-03

Table 6 Absolute error at $M = 2, k = 2$ with different α of Problem 5.2

z	$\alpha=0.25$	$\alpha=0.5$	$\alpha=0.75$	$\alpha=0.85$
0.125	5.683e-15	5.6777e-15	1.0804e-14	1.4573e-14
0.375	5.6934e-15	5.5962e-15	1.0398e-14	1.3845e-14
0.625	4.5155e-15	5.447e-15	1.19e-15	1.9203e-15
0.875	4.2154e-15	5.2753e-15	1.8111e-15	6.5746e-16

Table 7 Error comparison of Problem 5.3

$\hat{m}$	Present method	Triangular factorization method (Maleknejad et al. 2010)
4	8.2e-07	3.9e-2
8	5.4e-07	9.9e-3
16	5.3e-07	2.5e-3

Table 8 Error comparison of Problem 5.3

α	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$
$M = 2, k = 2$	3.6e-8	3.1e-7	8.6e-7
$M = 3, k = 2$	5.9e-9	8.4e-8	1.4e-7
$M = 4, k = 3$	2.2e-7	4.5e-7	2.7e-7

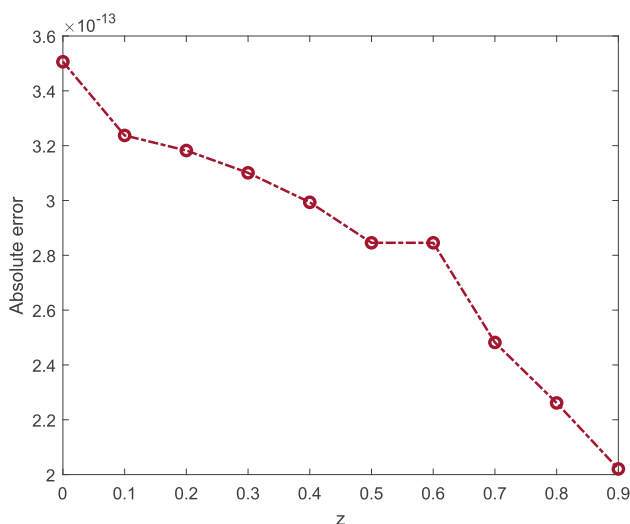


Fig. 4 At $M = 4$, $k = 2$ absolute error function plot for Problem 5.2

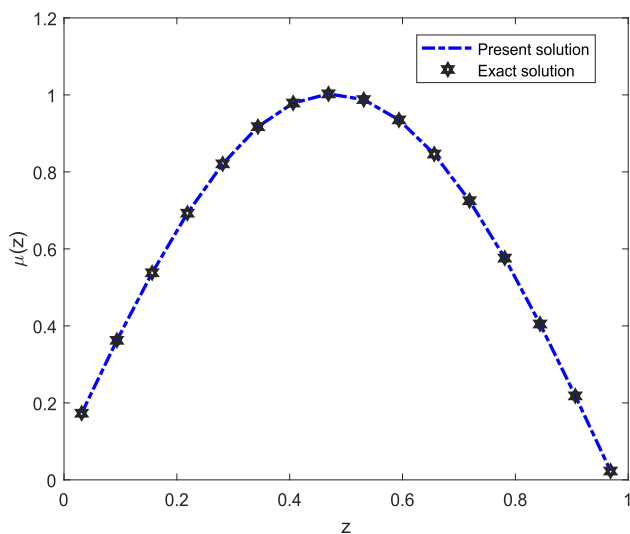


Fig. 5 Comparison of generalized Bell wavelet solution and exact solution for Problem 5.3

Test Problem 5.4 First, consider the linear VIE:

$$\mu(z) = \cos(z) + \int_0^z (z-t)\cos(z-t)\mu(t)dt. \quad (5.4)$$

The exact solution of (5.4) is $\mu(z) = \frac{1}{3}(2\cos(\sqrt{3}z) + 1)$. Figure 6 displays a comparison between the solution obtained using the generalized Bell wavelet and the exact solution. To demonstrate the effectiveness of the current method, a comparison of absolute errors is presented in Table 9. The present method is compared with the Bernoulli method (Shiralashetti

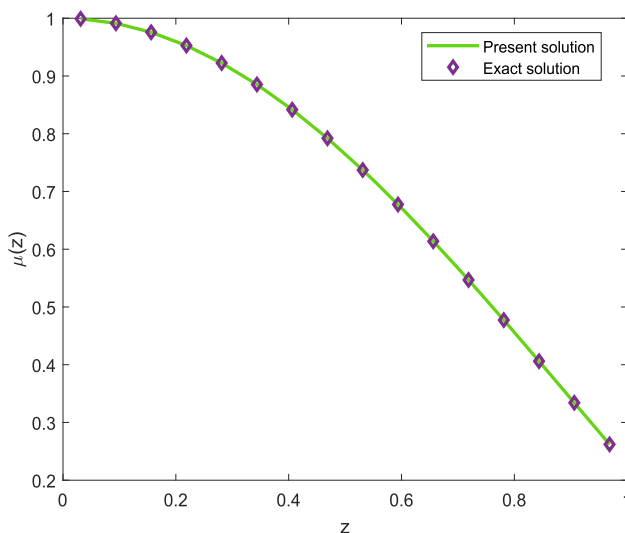


Fig. 6 Comparison of generalized Bell wavelet solution and exact solution for Problem 5.4

Table 9 Absolute error comparisons with of Problem 5.4

z	$M = 2, k = 2$		$M = 3, k = 2$	
	Present method	Bernoulli method (Shiralashetti et al. 2020)	Present method	Bernoulli method (Shiralashetti et al. 2020)
0.125	2.5345e−05	3.3929e−03	3.4621e−05	9.5507e−04
0.25	6.7649e−05	2.4228e−03	3.1441e−06	1.9812e−04
0.375	5.5448e−05	1.4527e−03	2.5396e−06	5.4593e−04
0.5	7.096e−05	1.8545e−03	1.7088e−06	1.4124e−03
0.625	1.7263e−05	2.2564e−03	7.3479e−06	1.6710e−03
0.75	8.3465e−05	2.2142e−03	5.5496e−05	1.9544e−04
0.875	1.0514e−05	2.3535e−03	2.8442e−06	1.6092e−03

et al. 2020) at $M = 2, k = 2$, and $M = 3, k = 2$. From the table and figure, it is clear that the present method provides better results for this problem.

Test Problem 5.5 Consider the following nonlinear VIE

$$\mu(z) = \frac{1}{9} + \frac{3}{4}x + \exp(z) - \frac{1}{9}\exp(3z) + \int_0^z (z-t)\mu^3(t) dt. \quad (5.5)$$

The exact solution of (5.5) is $\mu(z) = \exp(z)$. The operational matrices of integration are calculated to solve these types of integral equations. The comparison between the solution of the generalized Bell wavelet and the exact solution is given by Fig. 7. The results achieved by the current method at $M = 2, k = 2$, and $\alpha = 0.85$ are compared with the Fibonacci wavelet (Shiralashetti and Lamani 2021) at $k = 2, M = 3$. Table 10 shows the better performance of the present method compared to the Fibonacci method (Shiralashetti and Lamani 2021). The process takes 5.022 s of CPU time.

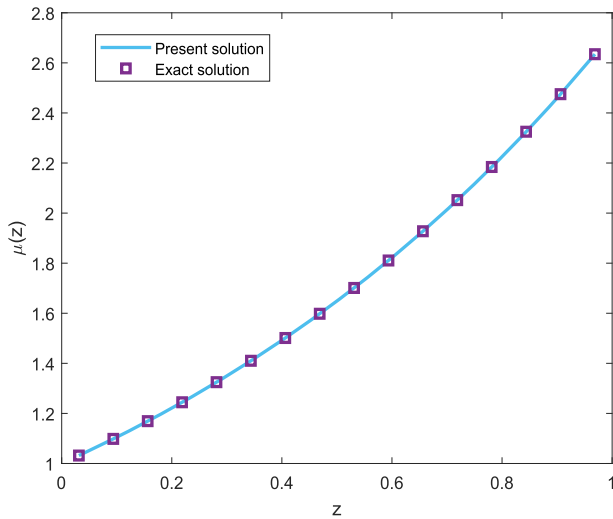


Fig. 7 Comparison of generalized Bell wavelet solution and exact solution for Problem 5.5

Table 10 Error comparison of Problem 5.5

z	Present method	Fibonacci method (Shiralashetti and Lamani 2021)
	$k = 2, M = 2, \alpha = 0.85$	$k = 2, M = 3$
0.08	3.7728e-06	1.6078e-03
0.25	5.4702e-06	5.3428e-04
0.42	9.1526e-06	4.7052e-03
0.58	1.6646e-06	8.0140e-03
0.75	3.1880e-06	1.2998e-03
0.92	6.4373e-06	1.9641e-02

Test Problem 5.6 Lastly, consider the following nonlinear VIE

$$\mu(z) = \frac{1}{2}(z - z^2) + \cos(z) - \sin(z) - \frac{1}{4}\sin(2z) + \int_0^z (z - t)\mu^2(t)dt. \quad (5.6)$$

The exact solution of (5.6) is $\mu(z) = \cos(z) - \sin(z)$. The comparison of solutions obtained by generalized Bell wavelet with exact at different collocation points is plotted in Fig. 8. The absolute error of the current approach at $M = 2, k = 2$, and $M = 3, k = 2$, is compared with the Fibonacci method (Shiralashetti and Lamani 2021) in Table 11. The present method has better performance throughout than the Fibonacci method (Shiralashetti and Lamani 2021). Additionally, the results give more improved accuracy with the increase of collocation points.

6 Conclusion

This paper introduced the generalized Bell wavelet collocation method as a solution for both linear and nonlinear integral equations, including FIE and VIE. The theoretical foundations of

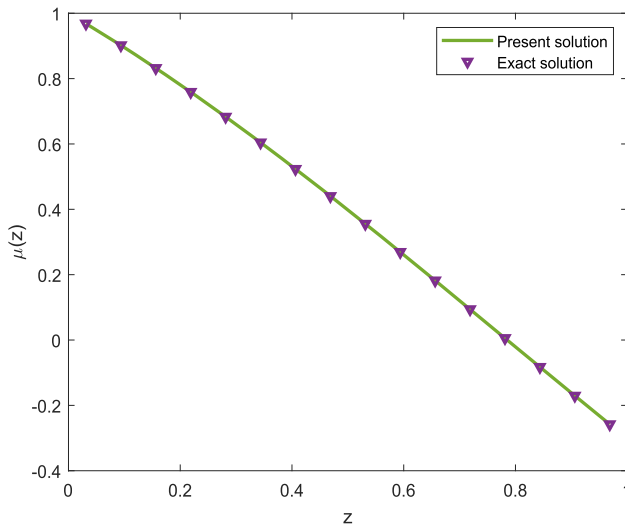


Fig. 8 Comparison of generalized Bell wavelet solution and exact solution for Problem 5.6

Table 11 Absolute errors for Problem 5.6

z	Present method		Fibonacci method (Shiralashetti and Lamani 2021)
	$M = 2, k = 2$	$M = 3, k = 2$	$M = 3, k = 2$
0.08	3.7677e-05	3.2843e-06	1.0577e-03
0.25	2.8863e-05	3.1109e-06	3.5367e-05
0.42	6.3209e-04	3.1856e-06	7.8662e-04
0.58	4.7193e-04	3.5225e-06	5.4614e-04
0.75	1.8705e-04	4.1613e-06	2.8998e-04
0.92	1.2773e-04	5.0668e-06	4.4093e-04

the algorithms were laid out, supported by numerical validation. A generalized Bell wavelet function approximation was provided. The operational matrices of integration were presented to approximate the VIEs. Firstly, the FIEs were of linear and nonlinear types. Figures 1, 3, and 5 presented the comparison of the proposed method with the exact solutions. Tables 1 and 6 provided details of the fractional parameter α at different points with resolution levels $M = 2, k = 2$, and $M = 3, k = 2$. Tables 2, 3, 4, 5, 6 and 7 showed the accuracy of this approach compared to the Fibonacci method (Yadav et al. 2023a), Hermite method (Maleknejad and Yousefi 2006), and triangular factorization method (Maleknejad et al. 2010) in terms of absolute and maximum absolute error. Secondly, the VIEs of linear and nonlinear types were solved. The absolute error of the proposed methodology was evaluated and compared with that of the Bernoulli method (Shiralashetti et al. 2020) and the Fibonacci method (Shiralashetti and Lamani 2021) in Tables 9, 10 and 11. It was evident from the outcomes that an increase in the value of k and M provided better results, and the parameter α yielded significant results in non-integer order.

In the future, the outlined methodology holds potential for expansion to address a broader range of mathematical challenges. This includes tackling the Fredholm-Volterra integral equations, Abel's integral equations, stochastic Abel's integral equations, and weakly singular Fredholm integral equations. Furthermore, leveraging the versatility of the generalized Bell wavelet presents opportunities for resolving fractional differential equations, such as the Bagley-Torvik equations and Riccati differential equations, among others.

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