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On nonclassical symmetries, Painlevé analysis and soliton solutions of three-coupled korteweg–de vries (KdV) system

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Abstract

The three coupled KdV system is investigated for exact solutions and Painlevé analysis. Exact solutions are examined through nonclassical symmetries via Bluman and Cole approach. Derived symmetries are generalizations of earlier obtained symmetries of the considered system. There is power series solution of the reduced ODEs of the examined system. Assuming the solutions in terms of Jacobi elliptic functions, some new soliton solutions of the system under consideration are obtained. These solutions are two-singular soliton, three-singular soliton, multi-soliton, multi-singular soliton, combined soliton, bright solion, dark soliton, and bell shaped soliton solutions. Further, graphical depiction of the exact solutions to the governing system. Using Kruskals method and symbolic software Maple, it is verified that the system has Painlevé property that represents integrability of the governing system.

1. Introduction

Nonlinear partial differential equations (NLPDEs) have great importance in investigating several physical models in various disciplines of engineering, technology, and science. Lie classical method was firstly introduced by Sophus Lie [1]. A suitable approach for obtaining invariant solutions of NLPDEs using classical symmetries of NLPDEs is the Lie classical method [2–4]. The KdV equation [5]

$$p_t + 6pp_y + p_{yyy} = 0, \tag{1.1}$$

has been employed to depict the long wavelength and small amplitude of shallow water waves in fluids, where p stands for the wave amplitude function for space and time coordinates y and t respectively. Nonlinear waves in a variety of physical fields including cosmic plasmas, oceans and planetary atmospheres have also been described by KdV-type equations [6–8].

Several coupled KdV systems have been derived from the Neumann system, that describes a particle on an N-dimensional sphere influenced by anisotropic harmonic quadratic potential [9, 10], where N is a natural number. It has been claimed that the KdV and the Boussinesq hierarchy are related to the Neumann system corrosive to the eigenvalue problems of the second and third orders, respectively [11, 12]. It has been considered that the Neumann system's fourth-order eigenvalue problem [13] is as follows

$$[\partial_y^4 + q(y,t)\partial_y^2 + \partial_y^2 q(y,t) + \iota\, p(y,t)\partial_y + \iota\, \partial_y p(y,t) + g(y,t)]z(y,t) = \vartheta z(y,t), \tag{1.2}$$

where, $\iota = \sqrt{-1}$, ϑ represents eigenvalue, $z(y, t)$ represents eigenfunction of eigenvalue problem (1.2) on ϑ. The eigenvalue problem (1.2)'s non-constant potential functions are $q(y, t)$, $p(y, t)$, and $g(y, t)$, based on the spatial coordinate y and temporal coordinate t. It has been established that the phase space of the Neumann system corrosive to eigenvalue problem (1.2), which is limited to a particular invariant manifold of finite dimensions, is analogous to a hierarchy of coupled KdV-type equations [13]. In such a hierarchy, the first member is as follows:

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$$
\begin{aligned}
p_t + \tfrac{3}{4}rp_y - \tfrac{3}{2}qq_y + \tfrac{1}{4}p_{yyy} &= 0,\\
q_t + \tfrac{3}{4}rq_y + \tfrac{3}{4}qr_y + \tfrac{1}{4}q_{yyy} &= 0,\\
r_t - \tfrac{3}{4}rr_y - \tfrac{3}{4}p_y - \tfrac{1}{8}r_{yyy} &= 0.
\end{aligned} \tag{1.3}
$$

The three coupled KdV system [14, 15] is an extension of the classical KdV equation, which is important for studying nonlinear waves and solitons. Due to the existence of numerous interacting components, many real-world systems cannot be adequately described by a single equation. As in fluid dynamics, you may have waves moving through a medium where various wave types interact. The three coupled KdV system useful for researchers across many fields because of its capacity to model intricate interactions, preserve integrability, and provide insights into soliton dynamics. This system of equations has been discussed by many authors for group-invariant solutions using symmetry reductions, conservation laws, and multi-soliton solutions by the Neumann system. Du *et al* [14] obtained invariant solutions with the help of Lie classical approach and established the system's nonlinear self-adjointness, which allowed them to present the conservation laws based on Lie point symmetries. Zuo *et al* [15] applied the Neumann system for finding the multi-soliton solutions and analyzed that the velocity and amplitude of each soliton remain invariant after the interaction, excluding phase transfer. Solitons are more interesting and important [16] from physical point of view. Here, we were able to get the generalized symmetries for system (1.3). Bluman and Cole propose the 'method of conditional symmetry', often known as the nonclassical symmetry method, which yields more generalized symmetries than earlier approaches. This technique makes use of invariant surface criteria in addition to Lie classical approach. This method provides set of determined nonlinear equations. Therefore, solving these nonlinear equations is a challenging undertaking. Numerous scholars [17–24] used this technique to solve different PDEs. As an instance of derived nonclassical symmetries, the Lie classical symmetries are also obtained. Nonclassical symmetries obtained in this study covers the classical symmetries of the considered system.

In mathematics and physics, it has long been an interesting and well-liked topic to find the exact solution of NLPDEs. Over the last few decades, their is a great significance of directly searching for solutions to NLPDEs. With computational tool like Maple and Matlab, we may now complete laborious and complex algebraic computations. By taking into account any infinitesimal generator, the system of NLPDEs can be transformed into a system of ordinary differential equations (ODEs), that can be solved using a variety of techniques, including Sardar sub-equation method [25, 26], extended auxiliary equation method [27], Jacobi elliptic function approach [28, 29], ansatz function approach [30], exponential rational function technique [31], modified exp-function method [32, 33], extended hyperbolic function method [34], enhanced algebraic method [35], unified solver method [36], Kudryashov approach [37], Fan sub-equation method [38], (G'/G) expansion method [39, 40], Bernoulli sub-equation function approach [41], etc. Research on solitons has grown in importance across a wide range of disciplines, such as fluid dynamics [42], optical fiber [43], and plasma physics [44]. In these domains, solitary waves are highly intriguing. The N-soliton solutions and intersoliton interactions of NLPDEs are obtained by applying Hirota bilinear method [45–48]. In order to investigate Hirota condition for the (2+1) combined equation, Wang *et al* [49] obtained N-soliton solutions.

In the field of science and engineering, solitons are amazing phenomena. Though the medium they move through usually tends to dissipate or disperse wave energy, they are recognizable wave-like frameworks that maintain their configuration and propagate without altering or depleting energy. Solitons are widely useful and have been discovered in many scientific and technical fields [50]. In nonlinear optics, solitons are used in optical fiber communication systems [51]. Their ability to transmit data over long distances without distortion makes them valuable for long distance optical communication [52]. In fluid dynamics, solitary waves can be viewed as either lone water waves in canals or tsunamis. These waves can travel over great distances with little energy loss [53]. Solitons can arise in plasmas [54] and essential for comprehending processes in space physics, fusion research [55], and many other fields [56, 57].

If all of the discrete singularities in a system of PDEs are integrals, or the simple poles, then the system has Painlevé property. The technique for analyzing the Painlevé property for PDEs was firstly proposed by Weiss *et al* [58, 59]. The Painlevé analysis of the governing system is carried out by using symbolic computational tool Maple and the Kruskal technique [60–63]. The Painlevé property provides information regarding the integrable behavior of NLPDEs [64, 65].

Motivation behind this study is that Lie classical symmetries of the considered system are not much generalized as compared to symmetries obtained by nonclassical approach. Further the integrable behavior of this system is not investigated earlier. So to determine exact solutions of the considered system using nonclassical approach and to study integrable behavior using Painlevé analysis, we have done this work. The obtained Jacobi elliptic solutions corresponding to reduced ODEs are in positive as well as negative powers of *sn*, *cn* and *dn* functions, are not obtained so far and very rare in literature. Although, neither nonclassical symmetries nor Painlevé analysis of the governing system (1.3) have been investigated, which represents novelty of the work. Authenticity of the solutions are justified through symbolic computational tool Maple except power series solutions. Power series solutions are verified manually. This work is structured as follows. In section 2, we perform symmetry analysis using nonclassical approach for system (1.3). In section 3, governing system is transformed into system of ODEs and exact solutions in terms of Jacobi elliptic functions are obtained. Section 4 does the Painlevé analysis of the system under consideration. Section 5 includes discussion of results with graphical interpretation. Conclusion is presented in section 6.

2. Nonclassical symmetries

Bluman and Cole [20] proposed the idea of nonclassical symmetry by analyzing the linear heat equation. The idea behind this approach is to extend system (1.3) by adding surface conditions that should be invariant, that are related to infinitesimal generator.

Assuming that the subsequent transformations in the Lie group are appropriate with the system (1.3)

$$\begin{aligned} y^* &= y + \omega\xi(y, t, p, q, r) + O(\omega)^2,\\ t^* &= t + \omega\rho(y, t, p, q, r) + O(\omega)^2,\\ p^* &= p + \omega\eta(y, t, p, q, r) + O(\omega)^2,\\ q^* &= q + \omega\phi(y, t, p, q, r) + O(\omega)^2,\\ r^* &= r + \omega\psi(y, t, p, q, r) + O(\omega)^2, \end{aligned} \tag{2.1}$$

where ω is any constant, and the infinitesimals corresponding to y, t, p, q and r are ξ, ρ, η, ϕ and ψ respectively.

For equation (2.1), infinitesimal generator is given by

$$V = \xi\frac{\partial}{\partial y} + \rho\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial p} + \phi\frac{\partial}{\partial q} + \psi\frac{\partial}{\partial r}. \tag{2.2}$$

To determine nonclassical symmetries of governing system along with invariant surface condition, we utilize Bluman and Cole approach [20]. Here, consider

$$\begin{aligned} \Xi_1 &= p_t + \tfrac{3}{4}rp_y - \tfrac{3}{4}qq_y + \tfrac{1}{4}p_{yyy} = 0,\\ \Xi_2 &= q_t + \tfrac{3}{4}rq_y + \tfrac{3}{4}qr_y + \tfrac{1}{4}q_{yyy} = 0,\\ \Xi_3 &= r_t - \tfrac{3}{4}rr_y - \tfrac{3}{4}p_y - \tfrac{1}{8}r_{yyy} = 0,\\ \Xi_4 &= \rho p_t + \xi p_y - \eta = 0,\\ \Xi_5 &= \rho q_t + \xi q_y - \phi = 0,\\ \Xi_6 &= \rho r_t + \xi r_y - \psi = 0, \end{aligned} \tag{2.3}$$

where Ξ_1, Ξ_2, Ξ_3 are equations of considered system, Ξ_4, Ξ_5, and Ξ_6 represent invariance surface conditions. Therefore, under the infinitesimal generator V, the considered system with invariant surface conditions should be invariant. Here, we examine two instances. (i) $\rho = 1$, and (ii) $\rho = 0$.

Case (i) $\rho = 1$

The invariant surface conditions in the above equation with $\rho = 1$ take the following forms

$$\begin{aligned} p_t + \xi p_y - \eta = 0,\\ q_t + \xi q_y - \phi = 0,\\ r_t + \xi r_y - \psi = 0. \end{aligned} \tag{2.4}$$

Now, remove p_t, q_t, and r_t from Ξ_1, Ξ_2, and Ξ_3 using equation (2.4).

$$\begin{aligned} \eta - \xi p_y + \tfrac{3}{4}rp_y - \tfrac{3}{4}qq_y + \tfrac{1}{4}p_{yyy} = 0,\\ \phi - \xi q_y + \tfrac{3}{4}rq_y + \tfrac{3}{4}qr_y + \tfrac{1}{4}q_{yyy} = 0,\\ \psi - \xi r_y - \tfrac{3}{4}rr_y - \tfrac{3}{4}p_y - \tfrac{1}{8}r_{yyy} = 0. \end{aligned} \tag{2.5}$$

The following set of equations results by imposing the third prolongation to equation (2.5) and eliminating p_{yyy}, q_{yyy}, and r_{yyy}.

$$\begin{aligned} &\xi_p = 0,\ \xi_q = 0,\ \xi_r = 0,\ \xi_{yy} = 0,\\ &\eta_{pp} = 0,\ \eta_{yp} = 0,\ \eta_q = 0,\ \eta_r = 0,\\ &\phi_{qq} = 0,\ \phi_{qy} = 0,\ \phi_p = 0,\ \phi_r = 0,\\ &\psi_{rr} = 0,\ \psi_{ry} = 0,\ \psi_q = 0,\ \psi_p = 0,\\ &\tfrac{3\xi_y}{2} + \tfrac{3\eta_p}{4} - \tfrac{3\eta_r}{4} = 0, \end{aligned}$$

$$\begin{aligned}\frac{3(\eta_p)q}{2} - 3(\xi_y)q - \frac{3\phi}{2} - \frac{3(\phi_q)q}{2} = 0,\\ \frac{3(\psi_r)q}{4} + \frac{3\phi}{4} - \frac{3(\phi_q)q}{4} + \frac{3(\xi_y)q}{2} = 0,\\ \xi_t + \frac{(6\xi - 3r)\xi_y}{2} - \frac{3\psi}{4} = 0,\\ \phi_t + \frac{3r(\phi_y)}{4} + 3(\xi_y)\phi + \frac{3q(\psi_y)}{4} = 0,\\ \eta_t + 3(\xi_y)\eta + \frac{3r(\eta_y)}{4} - \frac{3q(\phi_y)}{2} = 0,\\ \psi_t + 3(\xi_y)\psi - \frac{3(\psi_y)r}{4} - \frac{3\eta_y}{4} = 0.\end{aligned} \tag{2.6}$$

On solving the resulting determining equations, we acquire

$$\begin{aligned}\xi = \frac{y + a_1}{3t + a_2}, \quad \eta = \frac{-4p + a_3}{3t + a_2},\\ \phi = \frac{-3q}{3t + a_2}, \quad \psi = \frac{-2r}{3t + a_2}.\end{aligned} \tag{2.7}$$

In relation to these obtained infinitesimals, the infinitesimal generator is provided by

$$V_1 = \frac{\partial}{\partial t} + \frac{y + a_1}{3t + a_2}\frac{\partial}{\partial y} + \frac{-4p + a_3}{3t + a_2}\frac{\partial}{\partial p} + \frac{-3q}{3t + a_2}\frac{\partial}{\partial q} + \frac{-2r}{3t + a_2}\frac{\partial}{\partial r}. \tag{2.8}$$

The above equation can also be written as

$$V_1 = (3t + a_2)\frac{\partial}{\partial t} + (y + a_1)\frac{\partial}{\partial y} + (-4p + a_3)\frac{\partial}{\partial p} + (-3q)\frac{\partial}{\partial q} + (-2r)\frac{\partial}{\partial r}. \tag{2.9}$$

As per the Lie classical approach [14], the infinitesimals of system (1.3) are provided by

$$\begin{aligned}\xi = c_1 y + c_4, \quad \rho = 3c_1 t + c_2,\\ \eta = -4c_1 p + c_3, \quad \phi = -3c_1 q, \quad \psi = -2rc_1.\end{aligned} \tag{2.10}$$

For the previously obtained infinitesimals, the infinitesimal generator is provided by

$$V_1 = (3c_1 t + c_2)\frac{\partial}{\partial t} + (c_1 y + c_4)\frac{\partial}{\partial y} + (-4c_1 p + c_3)\frac{\partial}{\partial p} + (-3c_1 q)\frac{\partial}{\partial q} + (-2rc_1)\frac{\partial}{\partial r}. \tag{2.11}$$

It is easy to comprehend that the infinitesimal generator in equation (2.11) that was obtained through the classical approach is a particular instance of the infinitesimal generator in equation (2.9) with

$$a_1 = \frac{c_4}{c_1}, \quad a_2 = \frac{c_2}{c_1}, \quad a_3 = \frac{c_3}{c_1}.$$

Case (ii) $\rho = 0,\ \xi = 1$.

In this case, invariant surface conditions become

$$p_y = \eta, \quad q_y = \phi, \quad r_y = \psi. \tag{2.12}$$

The following system results from substituting the various derivatives of p, q, and r to zero.

$$\begin{aligned}&\eta_p = 0, \ \eta_q = 0, \ \eta_r = 0,\\ &\phi_p = 0, \ \phi_q = 0, \ \phi_r = 0,\\ &\psi_p = 0, \ \psi_q = 0, \ \psi_r = 0,\\ &\phi_t + \frac{3(\phi_y)r}{2} + \frac{3(\psi_y)q}{4} + \frac{\phi\psi}{2} + \frac{\phi_{yyy}}{4} = 0,\\ &\eta_t + \frac{3(\eta_y)r}{4} - \frac{3(\phi_y)q}{2} - \frac{3\eta\psi}{4} - \frac{3\phi^2}{2} + \frac{\eta_{yyy}}{4} = 0,\\ &\psi_t - \frac{3(\psi_y)r}{4} - \frac{3\eta_y}{4} - \frac{3\psi^2}{4} - \frac{\psi_{yyy}}{8} = 0.\end{aligned} \tag{2.13}$$

The following collection of infinitesimals are obtained after solving the above system

$$\begin{aligned}&(i) \quad \eta = 0, \quad \phi = 0, \quad \psi = \frac{4}{4a_4 - 3t},\\ &(ii) \quad \eta = a_5 t + a_6, \quad \phi = 0, \quad \psi = \frac{-4a_5}{3a_5 t + 3a_6},\\ &(iii) \quad \eta = \left(\frac{3(a_7 t + a_8)^4}{128a_7} + a_9\right)(a_7 t + a_8), \quad \phi = \frac{(a_7 t + a_8)^2}{4}, \quad \psi = \frac{-4a_7}{3a_7 t + 3a_8}.\end{aligned} \tag{2.14}$$

Symmetries obtained above are more generalized than the symmetries obtained in equation (2.10).

3. Exact solutions using symmetry reductions

Using the determined infinitesimal generators from preceding section, the symmetry analysis of the system under consideration is carried out here to obtain the reduced system of ODEs.

For V_1, the Lagrange's equations are as follows

$$\frac{dt}{3t + a_2} = \frac{dy}{y + a_1} = \frac{dp}{-4p + a_3} = \frac{dq}{-3q} = \frac{dr}{-2r}. \tag{3.1}$$

Following similarity variable and dependent variables were obtained by solving the above equations

$$\theta = \frac{3t + k_2}{3(y + k_1)^3}, \quad p(y, t) = \frac{k_3}{4} + \frac{P(\theta)}{(y + k_1)^4},$$
$$q(y, t) = \frac{Q(\theta)}{(y + k_1)^3}, \quad r(y, t) = \frac{R(\theta)}{(y + k_1)^2}. \tag{3.2}$$

where $P(\theta)$, $Q(\theta)$, and $R(\theta)$ are functions of similarity variable θ. After replacing dependent variables and their different derivatives from equation (3.2) in system (1.3), we get

$$\begin{aligned}
&-9\theta\dot{P}(\theta)R(\theta) + 4\dot{P}(\theta) - 384\theta\dot{P}(\theta) + 18Q(\theta)^2 + 18Q(\theta)\dot{Q}(\theta) - 12P(\theta)R(\theta)\\
&-120P(\theta) - 216\theta^2\ddot{P}(\theta) + \theta^3\dddot{P}(\theta) = 0,\\
&-9\theta\dot{Q}(\theta)R(\theta) + 4\dot{Q}(\theta) - 276\theta\dot{Q}(\theta) - 15R(\theta)Q(\theta) - 9\theta Q(\theta)\dot{R}(\theta) - 60Q(\theta)\\
&-189\theta^2\ddot{Q}(\theta) - 27\theta^3\dddot{Q}(\theta) = 0,\\
&18\theta R(\theta)\dot{R}(\theta) + 8\dot{R}(\theta) + 186\theta\dot{R}(\theta) + 12R(\theta)^2 + 24R(\theta) + 18\theta\dot{P}(\theta)\\
&+162\theta^2\ddot{R}(\theta) + 24P(\theta) + 27\theta^3\dddot{R}(\theta) = 0.
\end{aligned} \tag{3.3}$$

To obtain explicit solution in terms of power series for the above reduced system of ODEs, assume that $P(\theta)$, $Q(\theta)$, and $R(\theta)$ have the following particular forms

$$P(\theta) = \sum_{\varsigma=0}^{\infty} a_\varsigma \theta^\varsigma, \quad Q(\theta) = \sum_{\varsigma=0}^{\infty} b_\varsigma \theta^\varsigma, \quad R(\theta) = \sum_{\varsigma=0}^{\infty} c_\varsigma \theta^\varsigma \tag{3.4}$$

where a_ς, b_ς, c_ς for $\varsigma = 0, 1, 2, \ldots$ are the coefficients to be obtained. From equation (3.4), we have

$$\begin{aligned}
&\dot{P}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma a_\varsigma \theta^{\varsigma-1}, \; \dot{Q}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma b_\varsigma \theta^{\varsigma-1}, \; \dot{R}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma c_\varsigma \theta^{\varsigma-1},\\
&\ddot{P}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)a_\varsigma \theta^{\varsigma-2}, \; \ddot{Q}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)b_\varsigma \theta^{\varsigma-2}, \; \ddot{R}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)c_\varsigma \theta^{\varsigma-2},\\
&\dddot{P}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)(\varsigma-2)a_\varsigma \theta^{\varsigma-3}, \; \dddot{Q}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)(\varsigma-2)b_\varsigma \theta^{\varsigma-3},\\
&\dddot{R}(\theta) = \sum_{\varsigma=0}^{\infty} \varsigma(\varsigma-1)(\varsigma-2)c_\varsigma \theta^{\varsigma-3}.
\end{aligned} \tag{3.5}$$

Considering equations (3.4), (3.5), equation (3.3) becomes

$$\begin{aligned}
&-9\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma}(\varsigma-j)a_{\varsigma-j}c_j\theta^\varsigma + 4\sum_{\varsigma=0}^{\infty}(\varsigma+1)a_{\varsigma+1}\theta^\varsigma - 384\sum_{\varsigma=0}^{\infty}\varsigma a_\varsigma\theta^\varsigma + 18\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma} b_{\varsigma-j}b_j\theta^\varsigma\\
&-12\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma} a_{\varsigma-j}c_j\theta^\varsigma + 18\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma}(\varsigma+1-j)b_{\varsigma+1-j}b_j\theta^\varsigma - 120\sum_{\varsigma=0}^{\infty} a_\varsigma\theta^\varsigma - 216\sum_{\varsigma=0}^{\infty}\varsigma(\nu-1)a_\varsigma\theta^\varsigma\\
&+27\sum_{\varsigma=0}^{\infty}\varsigma(\varsigma-1)(\varsigma-2)a_\varsigma\theta^\varsigma = 0,\\
&-9\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma}(\varsigma-j)b_{\varsigma-j}c_j\theta^\varsigma + 4\sum_{\varsigma=0}^{\infty}(\varsigma+1)b_{\varsigma+1}\theta^\varsigma - 276\sum_{\varsigma=0}^{\infty}\varsigma b_\varsigma\theta^\varsigma - 15\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma} b_{\varsigma-j}c_j\theta^\varsigma\\
&-9\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma}(\varsigma-j)c_{\varsigma-j}b_j\theta^\varsigma - 60\sum_{\varsigma=0}^{\infty} b_\varsigma\theta^\varsigma - 189\sum_{\varsigma=0}^{\infty}\varsigma(\varsigma-1)b_\varsigma\theta^\varsigma - 27\sum_{\varsigma=0}^{\infty}\varsigma(\varsigma-1)(\varsigma-2)b_\varsigma\theta^\varsigma = 0,\\
&18\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma}(\varsigma-j)c_{\varsigma-j}c_j\theta^\varsigma + 8\sum_{\varsigma=0}^{\infty}(\varsigma+1)c_{\varsigma+1}\theta^\varsigma + 186\sum_{\varsigma=0}^{\infty}\varsigma c_\varsigma\theta^\varsigma + 12\sum_{\varsigma=0}^{\infty}\sum_{j=0}^{\varsigma} c_{\varsigma-j}c_j\theta^\varsigma\\
&+24\sum_{\varsigma=0}^{\infty} c_\varsigma\theta^\varsigma + 18\sum_{\varsigma=0}^{\infty}\varsigma a_\varsigma\theta^\varsigma + \sum_{\varsigma=0}^{\infty} a_\varsigma\theta^\varsigma + 162\sum_{\varsigma=0}^{\infty}\varsigma(\varsigma-1)c_\varsigma\theta^\varsigma + 27\sum_{\varsigma=0}^{\infty}\varsigma(\varsigma-1)(\varsigma-2)c_\varsigma\theta^\varsigma = 0.
\end{aligned} \tag{3.6}$$

Substituting $\varsigma = 0$ in equation (3.6), we have

$$\begin{aligned} &4a_1 + 18b_0^2 - 12a_0c_0 + 18b_1b_0 - 120a_0 = 0,\\ &4b_1 - 15b_0c_0 - 60b_0 = 0,\\ &8c_1 + 12c_0^2 + 24c_0 + 24a_0 = 0. \end{aligned} \tag{3.7}$$

On simplifying equation (3.7), we get

$$\begin{aligned} a_1 &= \tfrac{1}{4}(12a_0c_0 - 18b_1b_0 - b_0^2 + 120a_0),\\ b_1 &= \tfrac{1}{4}(15b_0c_0 + 60b_0),\\ c_1 &= \tfrac{1}{8}(-12c_0^2 - 24c_0 - 24a_0). \end{aligned} \tag{3.8}$$

For $\varsigma \geqslant 1$

$$\begin{aligned} a_{\varsigma+1} &= \frac{1}{4(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(9(\varsigma - j)a_{\varsigma-j}c_j - 18b_{\varsigma-j}b_j + 12a_{\varsigma-j}c_j - 18(\varsigma + 1 - j)b_{\varsigma+1-j}b_j)\\ &\quad + (384\,\varsigma + 120 + 216\,\varsigma(\varsigma - 1) - 27\,\varsigma(\varsigma - 1)(\varsigma - 2))a_\varsigma\Bigg],\\ b_{\varsigma+1} &= \frac{1}{4(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(9(\varsigma - j)b_{\varsigma-j}c_j + 15b_{\varsigma-j}c_j + 9(\varsigma - j)c_{\varsigma-j}b_j)\\ &\quad + (276\,\varsigma + 60 + \varsigma(\varsigma - 1) + 27\,\varsigma(\varsigma - 1)(\varsigma - 2))b_\varsigma\Bigg],\\ c_{\varsigma+1} &= -\frac{1}{8(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(18(\varsigma - j)c_{\varsigma-j}c_j + 12c_{\varsigma-j}c_j) + (24 + 18\,\varsigma)a_\varsigma\\ &\quad + (186\,\varsigma + 24 + 162\,\varsigma(\varsigma - 1) + 27\,\varsigma(\varsigma - 1)(\varsigma - 2))c_\varsigma\Bigg]. \end{aligned} \tag{3.9}$$

Upon replacing the values of a_1, b_1, c_1 and $a_{\varsigma+1}, b_{\varsigma+1}, c_{\varsigma+1}$ in equation (3.4), the derived solution of ODEs (3.3) provided by

$$\begin{aligned} P(\theta) &= a_0 + \tfrac{1}{4}(12a_0c_0 - 18b_1b_0 - b_0^2 + 120a_0)\theta\\ &\quad + \sum_{\varsigma=0}^{\infty}\frac{1}{4(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(9(\varsigma - j)a_{\varsigma-j}c_j - 18b_{\varsigma-j}b_j + 12a_{\varsigma-j}c_j - 18(\varsigma + 1 - j)b_{\varsigma+1-j}b_j)\\ &\quad + (384\,\varsigma + 120 + 216\,\varsigma(\varsigma - 1) - 27\,\varsigma(\varsigma - 1)(\varsigma - 2))a_\varsigma\Bigg]\theta^{\varsigma+1},\\ Q(\theta) &= b_0 + \tfrac{1}{4}(15b_0c_0 + 60b_0)\theta\\ &\quad + \sum_{\varsigma=0}^{\infty}\frac{1}{4(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(9(\varsigma - j)b_{\varsigma-j}c_j + 15b_{\varsigma-j}c_j + 9(\varsigma - j)c_{\varsigma-j}b_j)\\ &\quad + (276\,\varsigma + 60 + \varsigma(\varsigma - 1) + 27\,\varsigma(\varsigma - 1)(\varsigma - 2))b_\varsigma\Bigg]\theta^{\varsigma+1},\\ R(\theta) &= c_0 + \tfrac{1}{8}(-12c_0^2 - 24c_0 - 24a_0)\theta\\ &\quad + \sum_{\varsigma=0}^{\infty}\frac{-1}{8(\varsigma+1)}\Bigg[\sum_{j=0}^{\varsigma}(18(\varsigma - j)c_{\varsigma-j}c_j + 12c_{\varsigma-j}c_j) + (24 + 18\,\varsigma)a_\varsigma\\ &\quad + (186\,\varsigma + 24 + 162\,\varsigma(\varsigma - 1) + 27\,\varsigma(\varsigma - 1)(\varsigma - 2))c_\varsigma\Bigg]\theta^{\varsigma+1}. \end{aligned} \tag{3.10}$$

Further, Lagrange's equations for other derived infinitesimal generator are as follows

$$\frac{dt}{\lambda} = \frac{dy}{\mu} = \frac{dp}{0} = \frac{dq}{0} = \frac{dr}{0}. \tag{3.11}$$

New dependent variables and similarity variable are given by

$$\theta = \lambda y - \mu t,\ \ p(y, t) = \mathcal{P}(\theta),\ \ q(y, t) = \mathcal{Q}(\theta),\ \ r(y, t) = \mathcal{R}(\theta). \tag{3.12}$$

where $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and $\mathcal{R}(\theta)$ are functions of similarity variable θ. After putting above equations in system (1.3), we obtain

$$-\mu\dot{\mathcal{P}} + \frac{3\lambda\mathcal{R}\dot{\mathcal{P}}}{4} - \frac{3\lambda\mathcal{Q}\dot{\mathcal{Q}}}{2} + \frac{\lambda^3\dddot{\mathcal{P}}}{4} = 0,$$
$$-\mu\dot{\mathcal{Q}} + \frac{3\lambda\mathcal{R}\dot{\mathcal{Q}}}{4} + \frac{3\lambda\mathcal{Q}\dot{\mathcal{R}}}{4} + \frac{\lambda^3\dddot{\mathcal{Q}}}{4} = 0,$$
$$-\mu\dot{\mathcal{R}} - \frac{3\lambda\mathcal{R}\dot{\mathcal{R}}}{4} - \frac{3\lambda\dot{\mathcal{P}}}{4} - \frac{\lambda^3\dddot{\mathcal{R}}}{8} = 0. \tag{3.13}$$

Let us now examine the particular solutions of equation (3.13) in terms of Jacobi elliptic functions.

$$\mathcal{P}(\theta) = a_0 + a_1 sn(\theta, \upsilon) + a_2 sn^2(\theta, \upsilon) + \frac{a_3}{sn(\theta,\upsilon)} + \frac{a_4}{sn^2(\theta,\upsilon)},$$
$$\mathcal{Q}(\theta) = b_0 + b_1 sn(\theta, \upsilon) + b_2 sn^2(\theta, \upsilon) + \frac{b_3}{sn(\theta,\upsilon)} + \frac{b_4}{sn^2(\theta,\upsilon)},$$
$$\mathcal{R}(\theta) = c_0 + c_1 sn(\theta, \upsilon) + c_2 sn^2(\theta, \upsilon) + \frac{c_3}{sn(\theta,\upsilon)} + \frac{c_4}{sn^2(\theta,\upsilon)}. \tag{3.14}$$

By replacing these values of $\mathcal{P}(\theta)$, $\mathcal{Q}(\theta)$, and $\mathcal{R}(\theta)$ in equation (3.13), after comparing the like power of $sn(\theta, \upsilon)$ and interpreting the derived system of determining equations, we acquire

Case 1:

$$\lambda = \lambda,\ \upsilon = \upsilon,\ \mu = \frac{-6\ \lambda^4 c_0 + 4\ \lambda^6(\upsilon^2+1) + 3\ b_4{}^2}{8\lambda^3},$$
$$a_0 = a_0,\ a_1 = 0,\ a_2 = \frac{\upsilon^2 b_4{}^2}{\lambda^2},\ a_3 = 0,\ a_4 = \frac{b_4{}^2}{\lambda^2},$$
$$b_0 = -\frac{b_4(b_4{}^2 + 4\ \lambda^6(1+\upsilon^2) - 4\ \lambda^4 c_0)}{4\lambda^6},\ b_1 = 0,\ b_2 = \upsilon^2 b_4,\ b_3 = 0,\ b_4 = b_4,$$
$$c_0 = c_0,\ c_1 = 0,\ c_2 = -2\ \lambda^2\upsilon^2,\ c_3 = 0,\ c_4 = -2\ \lambda^2. \tag{3.15}$$

By using equation (3.15) into equation (3.14), we have

$$\mathcal{P}(\theta) = a_0 + \frac{\upsilon^2 b_4{}^2 sn^2(\theta,\upsilon)}{\lambda^2} + \frac{b_4{}^2}{\lambda^2 sn^2(\theta,\upsilon)},$$
$$\mathcal{Q}(\theta) = -\frac{b_4(b_4{}^2 + 4\ \lambda^6(1+\upsilon^2) - 4\ \lambda^4 c_0)}{4\lambda^6} + b_4\upsilon^2 sn^2(\theta, \upsilon) + \frac{b_4}{sn^2(\theta,\upsilon)},$$
$$\mathcal{R}(\theta) = c_0 - 2\ \lambda^2\upsilon^2 sn^2(\theta, \upsilon) - \frac{2\lambda^2}{sn^2(\theta,\upsilon)}. \tag{3.16}$$

After changing the original variable back to above system, we determine the following solutions of the considered system

$$p(y, t) = a_0 + \frac{\upsilon^2 b_4{}^2 sn^2((\lambda y - \mu t),\upsilon)}{\lambda^2} + \frac{b_4{}^2}{\lambda^2 sn^2((\lambda y - \mu t),\upsilon)},$$
$$q(y, t) = -\frac{b_4(b_4{}^2 + 4\ \lambda^6(1+\upsilon^2) - 4\ \lambda^4 c_0)}{4\lambda^6} + b_4\upsilon^2 sn^2((\lambda y - \mu t), \upsilon) + \frac{b_4}{sn^2((\lambda y - \mu t),\upsilon)},$$
$$r(y, t) = c_0 - 2\ \lambda^2\upsilon^2 sn^2((\lambda y - \mu t), \upsilon) - \frac{2\lambda^2}{sn^2((\lambda y - \mu t),\upsilon)}. \tag{3.17}$$

Case 2:

$$\mu = \mu,\ \lambda = \lambda,\ \upsilon = \upsilon,\ a_0 = a_0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = \frac{b_4{}^2}{\lambda^2},$$
$$b_0 = -\frac{b_4(-3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 + 4\ \mu + \lambda^3))}{12\lambda^6},\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = b_4,$$
$$c_0 = \frac{3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 - 2\ \mu + \lambda^3)}{6\lambda^4},\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = -2\lambda^2. \tag{3.18}$$

By using equation (3.18) into equation (3.14), we have

$$\mathcal{P}(\theta) = a_0 + \frac{b_4{}^2}{\lambda^2 sn^2(\theta,\upsilon)},$$
$$\mathcal{Q}(\theta) = -\frac{b_4(-3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 + 4\ \mu + \lambda^3))}{12\lambda^6} + \frac{b_4}{sn^2(\theta,\upsilon)},$$
$$\mathcal{R}(\theta) = \frac{3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 - 2\ \mu + \lambda^3)}{6\lambda^4} - \frac{2\lambda^2}{sn^2(\theta,\upsilon)}. \tag{3.19}$$

After changing the original variable back to above system, we determine the following solutions of the considered system

$$p(y, t) = a_0 + \frac{b_4{}^2}{\lambda^2 sn^2((\lambda y - \mu t),\upsilon)},$$
$$q(y, t) = -\frac{b_4(-3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 + 4\ \mu + \lambda^3))}{12\lambda^6} + \frac{b_4}{sn^2((\lambda y - \mu t),\upsilon)},$$
$$r(y, t) = \frac{3\ b_4{}^2 + 4\ \lambda^3(\lambda^3\upsilon^2 - 2\ \mu + \lambda^3)}{6\lambda^4} - \frac{2\lambda^2}{sn^2((\lambda y - \mu t),\upsilon)}. \tag{3.20}$$

In the similar way, the exact solutions can be derived using Jacobi *cn* and Jacobi *dn* functions.

• Jacobi cn type solutions:

$$p(y,\,t)=a_0-\frac{{b_2}^2cn^2((\lambda y-\mu t),\upsilon)}{\lambda^2\upsilon^2}-\frac{{b_2}^2(\upsilon^2-1)}{\lambda^2\upsilon^4cn^2((\lambda y-\mu t),\upsilon)},$$
$$q(y,\,t)=\frac{-b_2(8\,\lambda^6\upsilon^6-4\,\lambda^6\upsilon^4+4\,\lambda^4\upsilon^4c_0-{b_2}^2)}{4\lambda^6\upsilon^6}+b_2cn^2((\lambda y-\mu t),\,\upsilon)+\frac{b_2(\upsilon^2-1)}{\upsilon^2cn^2((\lambda y-\mu t),\upsilon)},$$
$$r(y,\,t)=c_0+2\,\lambda^2\upsilon^2cn^2((\lambda y-\mu t),\,\upsilon)+\frac{2\,\lambda^2(\upsilon^2-1)}{cn^2((\lambda y-\mu t),\upsilon)}. \quad (3.21)$$

• Jacobi dn type solutions:

$$p(y,\,t)=a_0-\frac{{b_2}^2dn^2((\lambda y-\mu t),\upsilon)}{\lambda^2}+\frac{{b_2}^2(\upsilon^2-1)}{\lambda^2dn^2((\lambda y-\mu t),\upsilon)},$$
$$q(y,\,t)=\frac{b_2(4\,\lambda^6\upsilon^2-8\,\lambda^6-4\,\lambda^4c_0+{b_2}^2)}{4\lambda^6}+b_2\,dn^2((\lambda y-\mu t),\,\upsilon)+\frac{b_2(1-\upsilon^2)}{dn^2((\lambda y-\mu t),\upsilon)},$$
$$r(y,\,t)=c_0+2\,\lambda^2dn^2((\lambda y-\mu t),\,\upsilon)+\frac{2\,\lambda^2(1-\upsilon^2)}{dn^2((\lambda y-\mu t),\upsilon)}. \quad (3.22)$$

Various soliton solutions of governing system (1.3) represented by equations (3.17), (3.20), (3.21), and (3.22) in terms of Jacobi elliptic functions.

Here, $sn=\text{JacobiSN}(\theta,\upsilon)$, $cn=\text{JacobiCN}(\theta,\upsilon)$, $dn=\text{JacobiDN}(\theta,\upsilon)$, are the Jacobi elliptic functions with the modulus $0<\upsilon<1$.

• Hyperbolic functions are degenerated, when $\upsilon\longrightarrow 1$ such as

$$sn(\theta,\,1)=\tanh(\theta),\;\; cn(\theta,\,1)=\text{sech}(\theta),\;\; dn(\theta,\,1)=\text{sech}(\theta).$$

In this case, there are various type of soliton solutions, such as singular, dark, bright, bell-shaped, etc.

• Trigonometric functions are degenerated, when $\upsilon\longrightarrow 0$ such as

$$sn(\theta,\,0)=\sin(\theta),\;\; cn(\theta,\,0)=\cos(\theta),\;\; dn(\theta,\,0)=1.$$

In this case, the solutions are periodic. If we choose value of υ other than 0 and 1 then we have some different type of soliton solutions.

Singular soliton type solutions are due to negative powers of Jacobi elliptic functions sn, cn and dn, as by selecting some parameters which makes them zero in denominator.

4. Painlevé analysis

In order to verify whether the governing system exhibits integrable behavior, a Painlevé analysis is conducted on system (1.3).

In the vicinity of the singular manifolds $\Psi(y,t)=0$, one can assume that the Laurent expansions of the $p(y,t)$, $q(y,t)$, and $r(y,t)$ are

$$p(y,\,t)=[\Psi(y,\,t)]^{\delta_1}\sum_{n=0}^{\infty}p_n(y,\,t)\Psi^n(y,\,t),$$
$$q(y,\,t)=[\Psi(y,\,t)]^{\delta_2}\sum_{n=0}^{\infty}q_n(y,\,t)\Psi^n(y,\,t),$$
$$r(y,\,t)=[\Psi(y,\,t)]^{\delta_3}\sum_{n=0}^{\infty}r_n(y,\,t)\Psi^n(y), \quad (4.1)$$

where δ_1, δ_2 and δ_3 are integers and $\Psi(y,t)$ is an analytic function of (y,t). We formulate the following assumption for the leading order analysis

$$p(y,\,t)=p_0(y,\,t)[\Psi(y,\,t)]^{\delta_1},$$
$$q(y,\,t)=q_0(y,\,t)[\Psi(y,\,t)]^{\delta_2},$$
$$r(y,\,t)=r_0(y,\,t)[\Psi(y,\,t)]^{\delta_3}. \quad (4.2)$$

Putting above equations into considered system, and then balancing, we get the branch with leading order analysis as follows

$$\delta_1=-2,\;\delta_2=-2,\;\delta_3=-2,$$
$$p_0(y,\,t)=\frac{[q_0(y,t)]^2}{[\Psi_y(y,t)]^2},\;\; q_0(y,\,t)=q_0(y,\,t),\;\; r_0(y,\,t)=-2[\Psi_y(y,\,t)]^2, \quad (4.3)$$

where $q_0(y,t)$ is any function.

The resonance points of this branch are now found by substituting the following expressions into system (1.3)

$$
\begin{aligned}
p(y,t) &= p_0(y,t)\Psi^{-2} + p_n(y,t)\Psi^n,\\
q(y,t) &= q_0(y,t)\Psi^{-2} + q_n(y,t)\Psi^n,\\
r(y,t) &= r_0(y,t)\Psi^{-2} + r_n(y,t)\Psi^n.
\end{aligned} \tag{4.4}
$$

Once the highest singular term is balanced, the obtained resonance points are as follows

$$
s = -1, 0, 1, 2, 4, 5, 6. \tag{4.5}
$$

For the resonance found in equation (4.5), the Laurent expansion is

$$
\begin{aligned}
p(y,t) &= \tfrac{p_0}{\Psi^2} + \tfrac{p_1}{\Psi} + p_2 + p_3\Psi + p_4\Psi^2 + p_5\Psi^3 + p_6\Psi^4,\\
q(y,t) &= \tfrac{q_0}{\Psi^2} + \tfrac{q_1}{\Psi} + q_2 + q_3\Psi + q_4\Psi^2 + q_5\Psi^3 + q_6\Psi^4,\\
r(y,t) &= \tfrac{r_0}{\Psi^2} + \tfrac{r_1}{\Psi} + r_2 + r_3\Psi + r_4\Psi^2 + r_5\Psi^3 + r_6\Psi^4.
\end{aligned} \tag{4.6}
$$

By substituting equation (4.6) into system (1.3) and applying the Kruskal method, we have

$$
\begin{aligned}
&\Psi = y - f(t), \quad p_n = p_n(t),\\
&q_n = q_n(t), \quad r_n = r_n(t),
\end{aligned} \tag{4.7}
$$

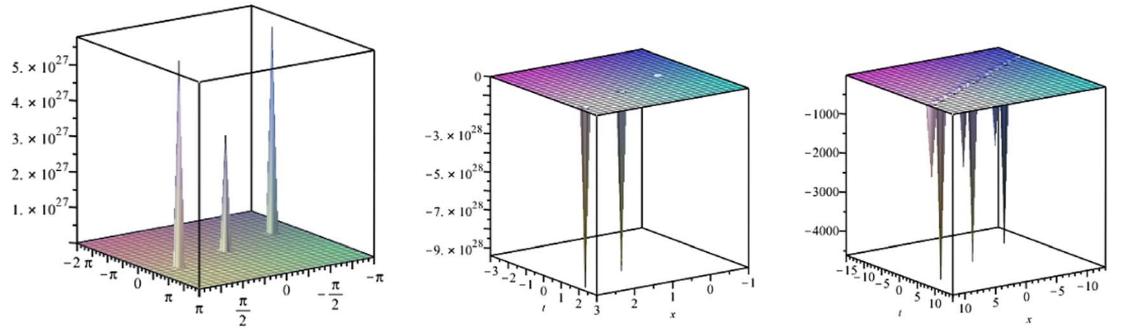
$f(t)$ represents any function that depends on t.

The following coefficient functions are obtained by substituting equations (4.6) and (4.7) into system (1.3),

$$
\begin{aligned}
&p_0(t) = q_0(t)^2, \ q_0(t) = q_0(t), \ r_0(t) = -2,\\
&p_1(t) = p_1(t), \qquad q_1(t) = 0, \qquad r_1(t) = 0,\\
&p_2(t) = p_2(t), \ q_2(t) = \tfrac{q_0(t)^3}{4} - \tfrac{4q_0(t)\dot{f}(t)}{3}, \ r_2(t) = \tfrac{q_0(t)^2}{2} - \tfrac{4\dot{f}(t)}{3},\\
&p_3(t) = -\tfrac{p_1(t)q_0(t)^2}{2} + \tfrac{2q_0(t)\dot{q}_0(t)}{3} + \tfrac{4p_1(t)\dot{f}(t)}{3},\\
&q_3(t) = \tfrac{p_1(t)q_0(t)}{4} - \tfrac{2\dot{q}_0}{3}, \qquad r_3(t) = \tfrac{p_1}{2},\\
&p_4(t) = -\tfrac{r_4(t)q_0(t)^2}{2} - \tfrac{p_1(t)^2}{8} + \tfrac{\dot{p}_1}{3}, \qquad q_4(t) = q_4(t), \ r_4(t) = r_4,\\
&p_5(t) = \tfrac{q_0(t)^4 p_1(t)}{32} + \tfrac{q_0^3(t)\dot{q}_0(t)}{2} - \tfrac{8q_0^2\ddot{f}(t)}{9} - \tfrac{8\dot{f}(t)q_0(t)\dot{q}_0(t)}{9} - \tfrac{8p_1(t)\dot{f}(t)^2}{9}\\
&\qquad - \tfrac{r_4(t)p_1(t)}{4} - \tfrac{q_0(t)q_5(t)}{2} + \tfrac{\dot{p}_2(t)}{3}, \qquad q_5(t) = q_5(t),\\
&r_5(t) = -\tfrac{p_1(t)q_0(t)^2}{4} - \tfrac{2q_0(t)\dot{q}_0(t)}{3} + \tfrac{4p_1(t)\dot{f}(t)}{3} + \tfrac{16\ddot{f}(t)}{9},\\
&p_6(t) = p_6(t), \qquad q_6 = -\tfrac{q_0(t)^3 r_4(t)}{8} + \tfrac{2\dot{f}(t)r_4(t)q_0(t)}{3} - \tfrac{p_1(t)^2 q_0(t)}{16} - \tfrac{q_4(t)q_0(t)^2}{4}\\
&\qquad - \tfrac{r_6(t)q_0(t)}{2} + \tfrac{p_1(t)\dot{q}_0(t)}{12} + \tfrac{2q_4(t)\dot{f}(t)}{3} - \tfrac{\dot{p}_1(t)q_0(t)}{12} + \tfrac{2\ddot{q}_0}{9}, \qquad r_6(t) = r_6(t).
\end{aligned} \tag{4.8}
$$

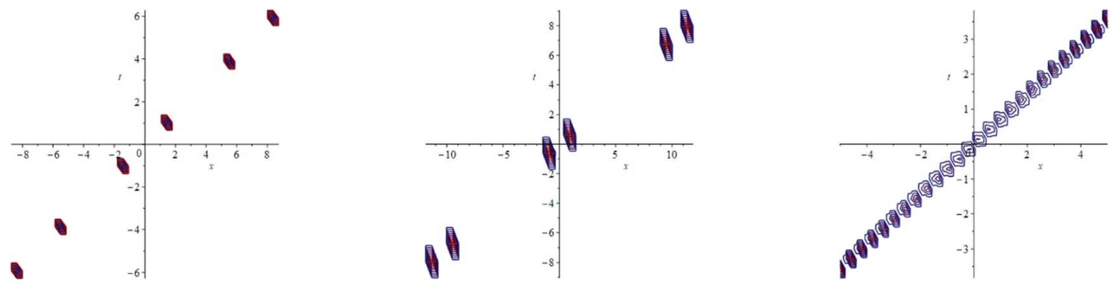
5. Discussion of results

In this work, the exact solutions of the examined system in terms of power series and Jacobi elliptic functions are obtained using nonclassical symmetry analysis. Also we investigate about integrability through Painlevé analysis using Kruskal approach. Using nonclassical approach, we get more generalized symmetries and obtained Jacobi elliptic solutions in terms of *sn*, *cn* and *dn*. The obtained solutions have positive as well as negative powers of *sn*, *cn* and *dn* functions, are not obtained so far and very rare in literature, Painlevé analysis has not yet done for the considered system, which shows the novelty of this work. One of the great physical significance of the Painlevé analysis is that it demonstrate the integrable behavior. In many disciplines, including mathematics, physics, and engineering, Jacobi elliptic functions are crucial, especially those involving integrals and differential equations. Their importance comes from their applications in complex analysis, specifically in the solution of differential equations and integrals that are beyond the scope of conventional trigonometric and hyperbolic functions techniques. The physical dynamics of systems in the real world can be best understood through visual representation. So we will discuss about graphical behavior of the derived solutions. By selecting the appropriate parameter values, we have created 3-D and contour plots. For $\lambda = 1.2, \upsilon = 1, a_0 = 2, a_2 = 1.6, a_4 = 1.6, b_0 = 2.6, b_2 = -1.5, b_4 = -1.5, \mu = 1.7, c_0 = 0.6, c_2 = -2.9, c_4 = -2.9$, figure 1 and figure 2 show 3-D and contour plots of three soliton, two-singular and multi-soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of equation (3.17). For $a_0 = -2.5, b_4 = 1.3, \lambda = 2, \upsilon = 1, \mu = 1$, Figures 5 and 4 show two-soliton, multi-soliton and combined soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of equation (3.20). For $a_0 = -2.5, b_2 = 3.2, \lambda = 2, \upsilon = 1, c_0 = 1.5, \mu = -5.7$, figure 5 and figure 6 display 3-D and contour plots of dark, bright, and bell shaped soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of



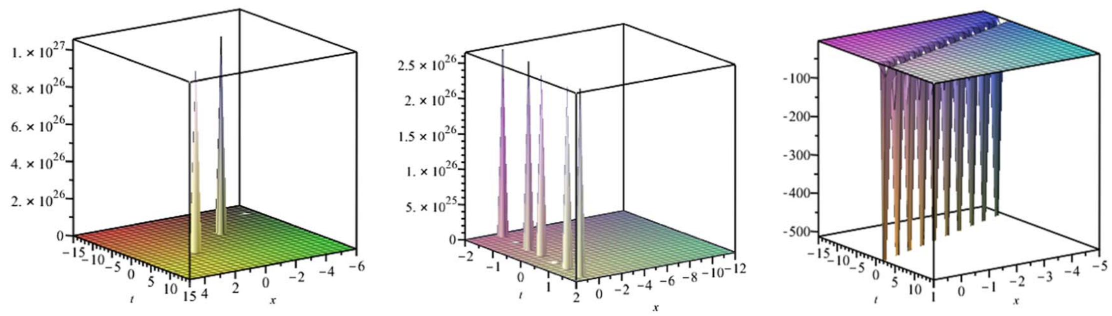
(a) $p(y, t)$ (b) $q(y, t)$ (c) $r(y, t)$

Figure 1. 3-D plots of three-singular soliton, two-singular and multi-soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of equation (3.17) with $\lambda = 1.2$, $\upsilon = 1$, $a_0 = 2$, $a_2 = 1.6$, $a_4 = 1.6$, $b_0 = 2.6$, $b_2 = -1.5$, $b_4 = -1.5$, $\mu = 1.7$, $c_0 = 0.6$, $c_2 = -2.9$, $c_4 = -2.9$.



(a) $p(y, t)$ (b) $q(y, t)$ (c) $r(y, t)$

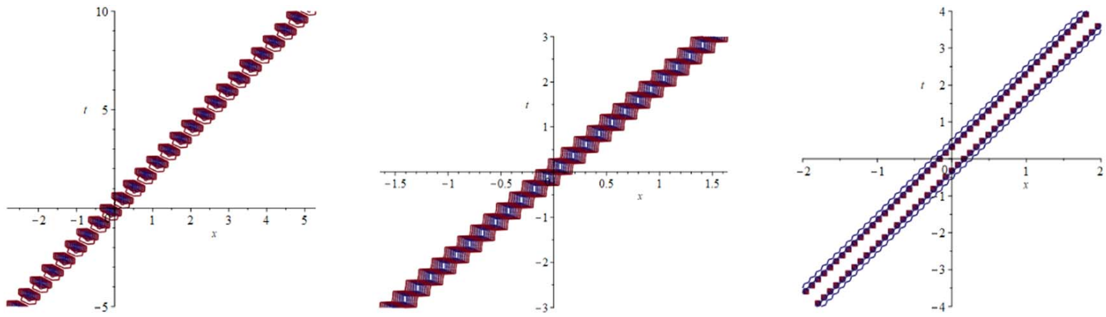
Figure 2. Contour plots of three-singular soliton, two-singular and multi-soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of equation (3.17) with $\lambda = 1.2$, $\upsilon = 1$, $a_0 = 2$, $a_2 = 1.6$, $a_4 = 1.6$, $b_0 = 2.6$, $b_2 = -1.5$, $b_4 = -1.5$, $\mu = 1.7$, $c_0 = 0.6$, $c_2 = -2.9$, $c_4 = -2.9$.



(a) $p(y, t)$ (b) $q(y, t)$ (c) $r(y, t)$

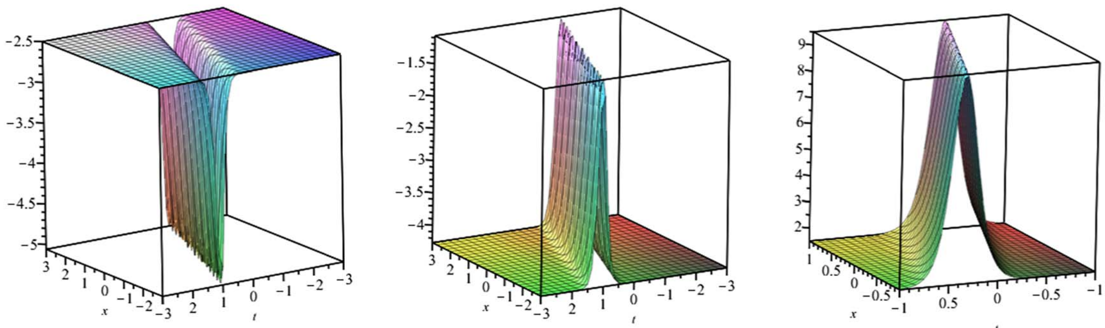
Figure 3. 3-D plots of two-singular soliton, multi-singular soliton and combined soliton solutions of $p(y, t)$, $q(y, t)$ and $r(y, t)$ respectively of equation (3.20) with $a_0 = -2.5$, $b_4 = 1.3$, $\lambda = 2$, $\upsilon = 1$, $\mu = 1$.

equation (3.21). Singular soliton type solutions are due to negative powers of Jacobi elliptic functions as by choosing some parameter values which makes them zero in denominator. In the similar way, we can plot 3-D and contour plots for equation (3.22). Solitons have been used to investigate numerous significant real-world issues in domains like nonlinear optics, fluid dynamics, plasma, molecular biology, and astrophysics. Today, due to the vast potential applications of solitons in information and communications technology make them one of the most active research topics. A continuous background field with localized areas of decreased and increased amplitude are represented by dark and bright soliton solutions respectively. Bell shaped soliton solutions are



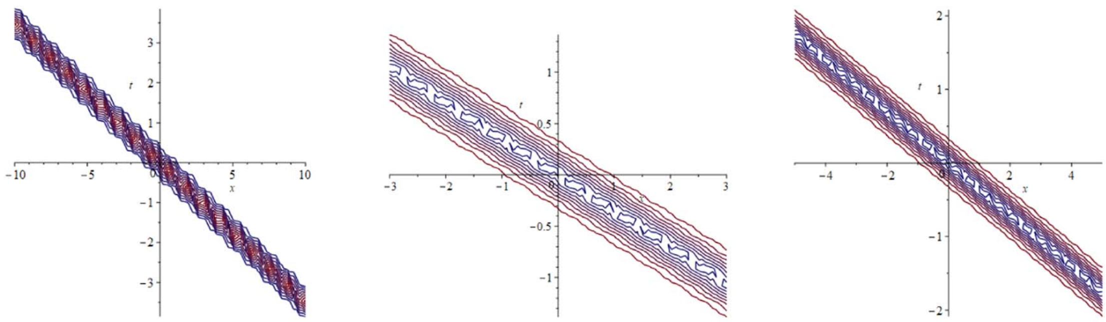
(a) $p(y,t)$ (b) $q(y,t)$ (c) $r(y,t)$

Figure 4. Contour plots of two-singular soliton, multi-singular soliton and combined soliton solutions of $p(y,t)$, $q(y,t)$ and $r(y,t)$ respectively of equation (3.20) with $a_0 = -2.5$, $b_4 = 1.3$, $\lambda = 2$, $\upsilon = 1$, $\mu = 1$.



(a) $p(y,t)$ (b) $q(y,t)$ (c) $r(y,t)$

Figure 5. 3-D plots of dark, bright and bell-shaped soliton solutions of $p(y,t)$, $q(y,t)$ and $r(y,t)$ respectively of equation (3.21) with $a_0 = -2.5$, $b_2 = 3.2$, $\lambda = 2$, $\upsilon = 1$, $c_0 = 1.5$, $\mu = -5.7$.



(a) $p(y,t)$ (b) $q(y,t)$ (c) $r(y,t)$

Figure 6. Contour plots of dark, bright and bell-shaped soliton solutions of $p(y,t)$, $q(y,t)$ and $r(y,t)$ respectively of equation (3.21) with $a_0 = -2.5$, $b_2 = 3.2$, $\lambda = 2$, $\upsilon = 1$, $c_0 = 1.5$, $\mu = -5.7$.

distinct from both dark and bright solitons, their bell-curve-like smooth, symmetrical, and usually peaked shape distinguishes them as a unique class of localized waveforms Singular soliton solutions are a particular kind of localized wave solution that has a singularity, which allows for the infinite value of some physical quantities (such as energy or amplitude) at a particular location in space or time. Multiple solitons coexisting and interacting in nonlinear systems are represented by multi-soliton solutions. Configurations in which various soliton types, such as dark and singular solitons, coexist and interact within the same system are referred to combined soliton solutions in nonlinear wave equations.

6. Conclusion

Nonclassical symmetries, reductions, and power series solutions that correspond to three coupled KdV system have been obtained and the exact solutions of governing system (1.3) in terms of Jacobi elliptic functions are determined. In one instance, the obtained infinitesimals exhibited more generalized symmetries than the classical one. Next, for some infinitesimals, the system under consideration is transformed into system of ODEs. The derived solutions exhibit the characteristics of various soliton solutions like two-singular soliton, three-singular soliton, combined soliton, bright, dark, bell shaped, multi-singular soliton and multi-soliton solutions. In mathematical physics and nonlinear dynamics, singular soliton solutions are fascinating objects having numerous applications in various fields such as plasma physics, fluid dynamics, optics, mathematical biology, etc. The self-focusing of laser beams, where the equilibrium between dispersion and nonlinearity results in stable pulse formations, can be modeled by singular solitons in nonlinear optics. Models of biological populations where singular behaviors result from nonlocal interactions or rapid changes, like in disease transmission or predator-prey dynamics, can make use of them. In section 4, we demonstrated the Painlevé integrability of system (1.3) by applying the Kruskal approach. The findings of this study clearly demonstrated that the nonclassical symmetry analysis and Kruskal method are useful tool that can be used to construct optical solitons and integrable behavior of the considered system. The obtained Jacobi elliptic solutions that correspond to reduced ODEs are in both positive and negative powers of the *sn*, *cn*, and *dn* functions. These solutions have not yet been obtained and are extremely uncommon in the literature. However, the work's novelty lies in the fact that neither nonclassical symmetries nor Painlevé analysis of the governing system (1.3) have been explored. In section 5, the results are discussed and presented with 3-D and contour plots. In future research directions, there is a possibility of examining approximate symmetries, nonclassical potential symmetries of three-coupled KdV system, which could produce more generalized symmetries. These generalized symmetries can then help make further reductions and make it possible to find new solutions related to these reductions. There is a possibility of examining space-time fractional three-coupled KdV system using symmetry analysis and other analytical techniques.

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Data availability statement

No new data were created or analysed in this study.

Contribution statement for CRediT authorship

Sharmila: Research, Original Writing Draft, Software, Methodology. Rajesh Kumar Gupta: Conceptualization, Visualization, Editing and Reviewing, Writing, Supervision.

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The authors assert that their personal relationships or known financial conflicts had no bearing on the findings of the study they presented in this work.

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References

[1] Lie S 1882 Uber die integration durch bestimmte integrable von einer Klasse linearer differentialgleichungen *Arch. Math.* **86** 3268
[2] Olver P 2000 *Applications of Lie Groups to Differential Equations* vol. 107 (Springer Science & Business Media)
[3] Bluman G W and Anco S C 2008 *Symmetry and Integration Methods for Differential Equations* vol. 154 (Springer Science & Business Media)
[4] Yadav S and Chauhan V 2024 Lie symmetry analysis, exact solutions and conservation laws of (3+1)-dimensional modified Wazwaz-Benjamin-Bona-Mahony equation *Phys Scr* **99** 095239
[5] Bhatta D D and Bhatti M I 2006 Numerical solution of KdV equation using modified Bernstein polynomials *Appl Math Comput* **174** 1255–68
[6] da Silva A G and da Rocha R 2017 Information-entropic analysis of Kortewegde Vries solitons in the quarkgluon plasma *Phys Lett B* **774** 98–102
[7] El-Tantawy S A and Moslem W M 2014 Nonlinear structures of the Korteweg-de Vries and modified Korteweg-de Vries equations in non-Maxwellian electron-positron-ion plasma: solitons collision and rogue waves *Phys Plasmas* **21** 052112
[8] Yang H, Yang D, Shi Y, Jin S and Yin B 2015 Interaction of algebraic Rossby solitary waves with topography and atmospheric blocking *Dyn Atmos Oceans* **71** 21–34
[9] Neumann C 1859 De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur *J. Reine Angew. Math* **56** 46–63
[10] Flaschka H and McLaughlin D W 1976 Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions *Prog Theor Phys* **55** 438–56
[11] Hirota R 2004 *The Direct Method in Soliton Theory* vol. 155 (Cambridge University Press)
[12] Gu Z 2002 The Neumann system for the 3rd-order eigenvalue problems related to the Boussinesq equation *Nuovo Cimento B Serie* **117** 615
[13] Zhao Y, Gu Z and Liu Y 2012 The Neumann system for the 4th-order eigenvalue problem and constraint flows of the coupled KdV-type equations *Eur Phys J Plus* **127** 1–14
[14] Du X X, Tian B, Yuan Y Q, Zhang C R and Du Z 2019 Symmetry reductions, group-invariant solutions and conservation laws of a three-coupled Korteweg-de Vries system *Chin J Phys* **60** 665–75
[15] Zuo D, Gao Y T, Meng G Q, Shen Y J and Yu X 2014 Multi-soliton solutions for the three-coupled KdV equations engendered by the Neumann system *Nonlinear Dynam* **75** 701–8
[16] Mathanaranjan T, Rajan M, Veni S S and Yildirim Y 2024 Cnoidal waves and solitons to three-coupled nonlinear Schrdinger's equation with spatially-dependent coefficients *Ukr J Phys Opt* **25** S1003–01016
[17] Arrigo D, Broadbridge P and Hill J 1993 Nonclassical symmetry solutions and the methods of Bluman-Cole and Clarkson-Kruskal *J Math Phys* **34** 4692–703
[18] Verma P and Kaur L 2020 Nonclassical symmetries and analytical solutions to Kawahara equation. *Int J Geom Methods Mod Phys* **17** 2050118
[19] Nucci M and Clarkson P 1992 The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation *Phys Lett A* **164** 49–56
[20] Bluman G W and Cole J D 1969 The general similarity solution of the heat equation *J Math Mech* **18** 10251042
[21] Estevez P 1994 Non-classical symmetries and the singular manifold method: the Burgers and Burgers-Huxley equations *J Phys A: Math Gen* **27** 2113
[22] Gupta R K and Singh M 2017 Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method *Nonlinear Dynam* **87** 1543–52
[23] Gulsen S, Hashemi M S, Alhefthi R, Inc M and Bicer H 2023 Nonclassical symmetry analysis and heir-equations of forced Burger equation with time variable coefficients *Comput Appl Math* **42** 221
[24] Srivastava S and Kumar M 2024 Nonclassical symmetries, optimal classification, and dynamical behavior of similarity solutions of (3+1)-dimensional Burgers equation *Chin J Phys* **89** 404–16
[25] Tariq K U, Bekir A, Nisar S and Alp M 2024 Construction of new wave structures and stability analysis for the nonlinear Klein-Gordon equation *Phys Scr* **99** 055220
[26] Chahlaoui Y, Butt A R, Abbas H and Bekir A 2024 Novel exact traveling wave solutions of Newton-Schrdinger system using Nucci reduction and Sardar sub-equation methods *Phys Scr* **99** 085227
[27] Mathanaranjan T and Myrzakulov R 2024 Integrable Akbota equation: conservation laws, optical soliton solutions and stability analysis *Opt and Quantum Electron* **56** 564
[28] Elsadany A A, Alshammari F S and Elboree M K 2024 Wave solutions in (3+ 1)-dimensional generalized fractional mKdV-ZK equation utilizing Jacobi elliptic functions *Phys Scr* **99** 055242
[29] Khan M I, Khan A and Farooq A 2024 Analyzing the Kuralay-II equation: bifurcation, chaos, and sensitivity insights through conformable derivative and Jacobi elliptic function expansion *Phys Scr* **99** 095210
[30] Wang K J and Li S 2024 Novel complexiton solutions to the new extended (3+ 1)-dimensional Boiti-Leon-Manna-Pempinelli equation for incompressible fluid *Europhys Lett* **146** 62003
[31] ALBaidani M M, Ali U and Ganie A H 2024 The closed-form solution by the exponential rational function method for the nonlinear variable-order fractional differential equations *Front Phys* **12** 1347636
[32] Shakeel M, Shah N A and Chung J D 2023 Application of modified exp-function method for strain wave equation for finding analytical solutions *Ain Shams Eng J* **14** 101883
[33] Shakeel M, Attaullah, Shah N A and Chung J D 2023 Modified exp-function method to find exact solutions of microtubules nonlinear dynamics models *Symmetry* **15** 360
[34] Rehman H U, Awan A U, Tag-ElDin E M, Alhazmi S E, Yassen M F and Haider R 2022 Extended hyperbolic function method for the (2+1)-dimensional nonlinear soliton equation *Results Phys* **40** 105802
[35] Arnous A H, Hashemi M S, Nisar K S, Shakeel M, Ahmad J, Ahmad I, Jan R, Ali A, Kapoor M and Shah N A 2024 Investigating solitary wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics *Results Phys* **57** 107369
[36] Chou D, Rehman H U, Haider R, Muhammad T and Li T L 2024 Analyzing optical soliton propagation in perturbed nonlinear Schrdinger equation: a multi-technique study *Optik* **302** 171714
[37] Wang K J, Wang G D, Shi F, Liu X L and Zhu H W 2024 Variational principle, Hamiltonian, bifurcation analysis, chaotic behaviors and the diverse solitary wave solutions of the simplified modified Camassa-Holm equation *Int J Geom Methods Mod Phys* (https://doi.org/10.1142/S0219887825500136)

[38] Song Y, Ma Y, Yang B and Wang Z 2024 Exact solutions and bifurcations for the (3+ 1)-dimensional generalized KdV-ZK equation *Phys Scr* **99** 075205

[39] Kol G R and Tabi C B 2011 Application of the (G'/G)-expansion method to nonlinear blood flow in large vessels. *Phys Scr* **83** 045803

[40] Sahoo S and Ray S S 2016 New solitary wave solutions of time-fractional coupled Jaulent-Miodek equation by using two reliable methods *Nonlinear Dynam* **85** 11671176

[41] Wang K J, Shi F, Li S and Xu P 2024 Dynamics of resonant soliton, novel hybrid interaction, complex N-soliton and the abundant wave solutions to the (2+ 1)-dimensional Boussinesq equation *Alex Eng J* **105** 485–95

[42] Ma Y L and Li B Q 2023 Dynamics of soliton resonances and soliton molecules for the AB system in two-layer fluids *Nonlinear Dynam* **111** 13327–41

[43] Mollenauer L F, Stolen R H and Gordon J P 1980 Experimental observation of picosecond pulse narrowing and solitons in optical fibers *Phys Rev lett* **45** 1095

[44] Ahmad J, Akram S, Noor K, Nadeem M, Bucur A and Alsayaad Y 2023 Soliton solutions of fractional extended nonlinear Schrdinger equation arising in plasma physics and nonlinear optical fiber *Sci Rep* **13** 10877

[45] Singh M 2016 New exact solutions for (3+1)-dimensional Jimbo-Miwa equation *Nonlinear Dynam* **84** 875880

[46] Hoque M F and Alshammari F S 2020 Higher-order rogue wave solutions of the Kadomtsev PetviashviliBenjanim Bona Mahony (KP-BBM) model via the Hirota-bilinear approach *Phys Scr* **95** 115215

[47] Wang K J, Shi F, Li S, Li G and Xu P 2025 Resonant Y-type soliton, interaction wave and other wave solutions to the (3+1)-dimensional shallow water wave equation *J Math Anal Appl* **542** 128792

[48] He G, Han Y, Xu T and Wang M 2024 Solitons, lump and interactional solutions of the (3+1)-dimensional BLMP equation in incompressible fluid *Phys Scr* **99** 085267

[49] Wang X and Zhang L L 2023 N-soliton solutions of Hirota-Satsuma coupled KdV equations with variable coefficients *Phys Scr* **98** 125207

[50] Jaradat I and Alquran M 2020 Construction of solitary two-wave solutions for a new two-mode version of the Zakharov-Kuznetsov equation *Mathematics* **8** 1127

[51] Alquran M, Jaradat I, Yusuf A and Sulaiman T A 2021 Heart-cusp and bell-shaped-cusp optical solitons for an extended two-mode version of the complex Hirota model: application in optics *Opt Quantum Electron* **53** 26

[52] Gnerhan H, Khodadad F S, Rezazadeh H and Khater M M A 2020 Exact optical solutions of the (2+1) dimensions Kundu-Mukherjee-Naskar model via the new extended direct algebraic method *Modern Phys Lett B* **34** 2050225

[53] Zhu W H, Pashrashid A, Adel W, Gunerhan H, Nisar K, Saleel C A, Inc M and Rezazadeh H 2021 Dynamical behaviour of the foam drainage equation *Results Phys* **30** 104844

[54] Raut S, Ma W X, Barman R and Roy S 2023 A non-autonomous Gardner equation and its integrability: solitons, positons and breathers *Chaos Solitons Fractals* **176** 114089

[55] Wang K J, Peng X and Shi F 2023 Nonlinear dynamic behaviors of the fractional (3+1)-dimensional modified Zakharov-Kuznetsov equation *Fractals* **31** 2350088

[56] Khater M and Ghanbari B 2021 On the solitary wave solutions and physical characterization of gas diffusion in a homogeneous medium via some efficient techniques *Eur Phys J Plus* **136** 128

[57] Az-Zobi E, Al-Maaitah A F and Osman M A T 2022 MS. New generalised cubic-quintic-septic NLSE and its optical solitons *Pramana* **96** 184

[58] Weiss J, Tabor M and Carnevale G 1983 The Painlev property for partial differential equations *J Math Phys* **24** 522–6

[59] Kumar V, Gupta R and Jiwari R 2013 Painlev analysis, Lie symmetries and exact solutions for variable coefficients Benjamin-Bona-Mahony-Burger (BBMB) equation *Commun Theor Phys* **60** 175

[60] Jimbo M, Kruskal M and Miwa T 1982 Painlev test for the self-dual Yang-Mills equation *Phys Lett A* **92** 59–60

[61] Hao X, Liu Y, Li Z and Ma W X 2019 Painlev analysis, soliton solutions and lump-type solutions of the (3+1)-dimensional generalized KP equation *Comput Math Appl* **77** 724–30

[62] Kumar S 1983 Painlev analysis and invariant solutions of Vakhnenko-Parkes (VP) equation with power law nonlinearity *Nonlinear Dynam* **24** 522–6

[63] Jyoti D and Kumar S 2020 Modified Vakhnenko-Parkes equation with power law nonlinearity: painlev analysis, analytic solutions and conservation laws *Eur Phys J Plus* **135** 1–12

[64] Fan L and Bao T 2022 Painlev integrability and superposition wave solutions of WhithamBroerKaup equations *Nonlinear Dynam* **109** 3091–100

[65] Fan L and Bao T 2024 Similarity wave solutions of WhithamBroerKaup equations in the oceanic shallow water *Phys Fluids* **36**