

# Role of the complexity factor and Karmarkar condition in constructing new wormhole models in dRGT gravity

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## ABSTRACT

This study delves into the distinctive characteristics of wormhole models in the context of de Rham-Gabadadze-Tolley (dRGT) massive gravity, providing insights into their theoretical behavior and stability. We use a null zero complexity factor to find the wormhole shape function for Model I. Additionally, we solve analytically the modified field equations describing wormhole for a given choice of logarithmic redshift function, exploiting the Karmarkar condition for embedding class one metrics for Model II. To achieve this, we analyze the wormhole geometry in a static spherical spacetime with an anisotropic matter configuration. The study investigates a number of parameters, including density, energy conditions, equation of state parameter, adiabatic sound velocity, and equilibrium condition. The solution shows a traversable wormhole that violates the null energy criterion and equilibrium state for certain ranges of free parameters. We employ adiabatic sound velocity analysis to concentrate on the stability of the wormhole. Furthermore, by using the equation of state parameter ( $\omega$ ), we conclude that both models end up in the phantom dark energy region. Finally, our findings highlight distinct photon deflection behaviors in dRGT massive gravity, with Model II showing negative angles indicative of repulsive gravity, while Model I exhibits positive angles, underscoring significant differences in gravitational dynamics.

## 1. Introduction

The spacetime geometry is elegantly described mathematically by the general theory of relativity. A black hole that is static and spherically symmetric is the result of one of the feasible solutions to Einstein's field equations Schwarzschild (1916). The existence of stellar-mass black holes in nature has been verified by the recent finding of gravitational waves (GWs) Abbott et al. (2016). It is interesting to note that Ludwig Flamm (1916) discovered in 1916 that another solution to Einstein's equations is now referred to as a white hole. It was thought that, in contrast to black holes, white holes expel light and matter from their event horizon. These two solutions might stand for two distinct spacetime regions connected by a conduit. Later, this conduit was referred to as a "bridge," and in 1935, Einstein and Rosen proposed the presence

of "bridges" through spacetime using the general theory of relativity Einstein and Rosen (1935). Several decades later, Misner and Wheeler were the first to introduce the term "wormhole" in reference Misner and Wheeler (1957). Ever since wormholes have been the subject of new research frontiers, but because they cannot be traversed, the original theory of wormholes was eventually disproved. This indicates how rapidly its throat opens and closes. However, a scalar field related to gravity can be added to stop the wormhole's throat from closing. This new component offers a more widespread class of wormholes, which was previously suggested by Bronnikov Bronnikov (1973) and Ellis Ellis (1973) on their own. Then some conditions were introduced by Morris and Thorne and revealed the possibility of wormholes and showed how they might be used to facilitate interstellar travel Morris et al. (1988). Theoretically, wormholes could exist according to Einstein's general theory of relativ-

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ity; however, this is still only speculation. On the other hand, modified gravity models add new phenomena or make adjustments to Einstein's general theory of relativity in order to solve some unanswered questions. The behavior and characteristics of wormholes may be predicted differently by these modifications than by general relativity. The predictions of modified gravity models for wormholes can differ based on the particular alterations performed and the presumptions involved, it is vital to remember this. Alternative explanations for wormholes or various characteristics and behaviors for these hypothetical structures may be suggested by certain modified gravity theories Mustafa et al. (2024b); Errehymy et al. (2024d); Errehymy (2024); Mustafa et al. (2023); Errehymy et al. (2023b,a); Battista et al.; Maurya et al. (2024); Errehymy et al. (2024a,b,e); Mustafa et al. (2024a). Wormholes are theoretically possible according to both Einstein's general theory of relativity and modified gravity models, perhaps their implications and predictions can differ. When comparing the wormhole predictions of modified gravity models with those derived from general relativity theory or other well-established models, there are both points of convergence and discrepancies Sadatian and Hosseini (2024). Discrepancies: In contrast to general relativity, wormholes in modified gravity models may remain stable without the need for exotic matter to prevent their collapse. Both general relativity and modified gravity theories recognize the potential existence of wormholes as theoretical concepts, even though their approaches to describing these phenomena may differ.

Massive gravity is one such theory that Pauli and Fierz developed Fierz and Pauli (1939). In particular, adding mass to the gravitational field through the inclusion of a spin-2 mass field in this theory is essential to comprehending the recent acceleration of expansion at vast ranges without the need for dark energy. Even before the Universe's accelerated expansion was discovered, research into enormous gravity was started. The authors Fierz and Pauli (1939) submitted a linear theory of massive gravity in 1939, considering the graviton as a mass term. Nevertheless, there were some limitations to this suggested model Zakharov (1970); van Dam and Veltman (1970) since the linear theory's asymptotic massless limits did not match the general theory of relativity predictions. A nonlinear technique within the framework of massive gravity, according to the author Vainshtein (1972), may provide a solution to this problem in 1972. However, using this approach results in an unfamiliar problem known as the Boulware-Deser ghost Hinterbichler (2012). Subsequently, the Boulle-Deser ghost was successfully ejected with the introduction of a new nonlinear form of massive gravity, called dRGT (de Rham, Gabadadze, and Tolley) de Rham et al. (2011). The dRGT formulation has now become well-known as a foundation for studying the cosmos at the cosmic scale. Furthermore, a number of fascinating articles have investigated the applications of dRGT massive gravity to unusual occurrences such as black holes Ghosh et al. (2016); Boonserm et al. (2018). The compact star, huge gravity, and dark energy star models are found in Li et al. (2023); Tudeschi et al. (2023) in dRGT. Additionally, the related charged black hole solution was discovered in ref. Berezhiani et al. (2012), and the spherically symmetric solutions for dRGT gravity were investigated in Brito et al. (2013); Nieuwenhuizen (2011).

The structure of the paper is as follows: First, we discussed about complexity terms and Karmarkar Condition in the introduction part. In section 2, we covered the background of the field equation of dRGT gravity. Thus we found the wormhole shape functions by using the null zero complexity approach and Karmarkar condition in section 3 and also covered the embedding diagram in the same part. Further, in section 4 we discuss the stability of the wormhole solution by using energy conditions, adiabatic sound velocity, equation of state parameters, and equilibrium condition. In the last, we concluded our results in section 6. Expressions for stress energy components are written in the appendix section.

### 1.1. Complexity terms

The simplest models are those that have zero complexity, such as the perfect crystal, which displays periodic behavior, and the isolated ideal gas, which displays random behavior. This is where the physics term “complexity” starts. A perfect crystal exhibits total order, with atoms organized according to precise symmetry rules. The perfect crystal's states have minimal information content, with a prevailing state of perfect symmetry near the center of the probability distribution. In contrast, the isolated ideal gas is completely disordered. The system has a maximum information potential and is equally likely to be found in any of its stages of accessibility. Given that these two fundamental systems “order” and “information” are at extremes, other elements must be taken into account when determining complexity. The “distance” from the equiprobable distribution of the system's accessible states is measured by the concept of “disequilibrium”, which was introduced. As a result, “disequilibrium” would be zero for the ideal gas and maximal for a perfect crystal as they are not uniformly distributed among the states that are accessible. A compromise is achieved between the concepts of “disequilibrium” and information by characterizing the complexity using a quantity that is a product of these two ideas. This eliminates the complexity of the perfect crystal and the ideal gas. A definition of complexity for self-gravitating systems has been proposed as a result of this idea. The complexity factor in a self-gravitating system was defined in 2018 by Herrera Herrera (2018). According to Herrera (2018); Herrera et al. (2018), one of the less complicated systems corresponds to a homogeneous fluid distribution with isotropic pressure, which is based on the fundamental assumption that complexity exists. Therefore, for such a distribution, the complexity factor is set to zero. Then, as an obvious candidate to measure the degree of complexity emerges a quantity that appears in the orthogonal splitting of the Riemann tensor, and that was denoted by  $y_{cf}$  and called the complexity factor. This proposal is mainly motivated by the fact that the scalar function  $y_{cf}$  contains contributions from both the local pressure anisotropy and the energy density inhomogeneity, combined in a very specific way, which disappears for the homogeneous and locally isotropic fluid distribution. It is noteworthy to mention that the complexity factor as defined disappears not only for the above-mentioned simple configuration, but it can also disappear when the two terms that appear in its definition- density inhomogeneity and anisotropic pressure-cancel one another.

A particular interaction between pressure anisotropy and energy density inhomogeneity gives rise to the complexity factor. It's interesting to note that when the effects of anisotropic pressure and non-uniform energy density balance each other out, systems with these two characteristics can still have a complexity factor of zero. In spherically symmetric static structures, Herrera's idea of complexity Herrera (2018) depends on pressure anisotropy and density inhomogeneity. There are two approaches to eliminate the complexity factor in static fluids: either the self-gravitating body's matter composition is made up of homogeneous density and isotropic pressure, or the pressure anisotropy cancels out the effects of density inhomogeneity. Therefore, finding precise solutions to a single generating function is made easier by the requirement that the complexity factor vanishes at every interior point of the star configuration. The gravitational behavior of the model can be fully calculated if a metric function is chosen based on physical considerations. For this condition, we now build a new bridge equation based on the vanishing complexity factor condition that connects metric functions. To establish the relation, we first use Herrera's concept of the complexity factor for a compact object Herrera (2018) to calculate the complexity factor ( $y_{cf}$ ) is defined as

$$y_{cf} = (P_r - P_t) - \frac{1}{2r^3} \int_0^r r^3 \frac{d\rho(r)}{dr} dr. \quad (1)$$

Substituting  $\rho$ ,  $P_r$  and  $P_t$  in the above equation, we get  $y_{cf}$ . Then employing the vanishing condition, i.e.,  $y_{cf} = 0$  and integrating we got the required relation between the metric potentials  $\zeta$  and  $R_s(r)$  as

$$e^{2\zeta(r)} = \left( A_1 \int r \left( 1 - \frac{R_s}{r} \right)^{-1/2} dr + B_1 \right)^2, \quad (2)$$

where  $A_1$  and  $B_1$  are denoted as the constants of integration.

The notion of complexity is currently employed in different gravitational theories to analyze the solutions to problems across various situations Andrade and Contreras (2021); Andrade et al. (2022); Contreras and Stuchlik (2022); Contreras et al. (2022); Maurya et al. (2022a,c, 2023a); Sharif and Hassan (2023); Sharif and Naseer (2023a); Zubair et al. (2023b,c). In this connection, Bhattacharya et al. considered a complexity factor parametrization and found viable traversable wormhole geometry Bhattacharya and Nalui (2023). Zubair et al. investigate Casimir wormhole solutions in Einstein Gauss-Bonnet gravity Zubair and Farooq (2023). Tayde et al. examine wormhole geometries in the context of  $f(Q, T)$  gravity under the background of non-commutative distributions with their complexity factor Tayde et al. (2024). Avalos et al. provide the redshift function of a Casimir traversable wormhole which, in combination with a non-vanishing complexity factor, leads to a traversable wormhole with a minimum amount of exotic matter Avalos et al. (2022).

### 1.2. Karmarkar condition

It was first demonstrated by Karmarkar Karmarkar (1948a) that a static spherically symmetric spacetime must have an embedding of one. It is referred to as the Karmarkar condition and is highly useful for determining the precise solutions of field equations. In essence, it supports finding the field equation solution for compact objects. The Einstein field equation (EFE) can be solved exactly by using the embedding class one condition, also known as the Karmarkar condition. In order to examine the basic geometric properties of objects, Riemann developed the idea known as Riemannian geometry. Schlaefli then estimates a Riemannian manifold with an algebraic metric and a positively defined signature to have the property of being locally and isometrically embedded inside a higher-dimensional flat Euclidean space. Embedding class one focuses on space-time geometry and imposes constraints based on whether it can be embedded in a higher-dimensional flat space. By reducing the number of degrees of freedom in the space-time equations, this constraint can make them simpler and frequently result in more manageable circumstances for solving them. In embedding class one condition there is a relation between metric functions and these do not depend on the field equations, also we can easily solve these metric functions. The Karmarkar condition Karmarkar (1948b) given by

$$e^{2\zeta(r)} = \left( \hat{A}_3 + \hat{B}_3 \int \sqrt{\left( 1 - \frac{R_s}{r} \right)^{-1} - 1} dr \right)^2, \quad (3)$$

where constants of integration are  $\hat{A}_3$  and  $\hat{B}_3$ .

Many researchers used this condition to discuss the physical properties of compact objects Kuhfittig (2018); Gedela et al. (2018); Errehymy et al. (2024a); Naseer et al. (2023). Mustafa et al. obtained two new embedded wormhole solutions by using the class-I approach in the background of newly fourth-order Ricci inverse gravity Mustafa et al. (2024a). Hassan et al. also find the wormhole solution in  $f(Q)$  gravity by using the Karmarkar condition and show that the combination of such ingredients provides us the possibility of obtaining traversable wormholes satisfying the energy conditions Mustafa et al. (2021). Chaudhary et al. investigated a wormhole solution by using radial dependent red shift function with Karmarkar condition Chaudhary et al. (2024). Furthermore, in some earlier works Malik et al. (2022); Kuhfittig (2018); Malik et al. (2024); Sharif and Fatima (2023), the authors use the Karmarkar condition by assuming the variable redshift function to generate

the shape functions of the wormholes. In the current work, the shape function of the wormhole is obtained using the Karmarkar condition (please refer to section 3.2 for more details).

Apart from the above two conditions, another two conditions describe the relationship between the metric functions. The conformally flat condition can be also used to solve the EFE that provides a relationship between the metric functions. In this regard, Ivanov Ivanov (2018) proposed a relation which is related to conformally flat geometry Stewart (1982); Herrera et al. (2001) given by

$$e^{2\zeta(r)} = \hat{A}_1^2 r^2 \cosh^2 \left( \int \sqrt{\frac{(1 - \frac{R_s}{r})^{-1}}{r^2}} dr + \hat{B}_1 \right), \quad (4)$$

where the constants are  $\hat{A}_1$  and  $\hat{B}_1$ .

Another different approach to obtain the solution to EFE was developed within the framework of a conformal killing vector, also referred to as a conformal symmetry condition. In the presence of Conformal Killing vectors, constants of motion along null geodesics are generated. A conformal symmetry places restrictions on the metric functions and frequently results in new exact solutions. The second relation Herrera et al. (1984); Herrera and Ponce de Leon (1985); Rahaman et al. (2017); Singh et al. (2018) subject to a conformal killing vector is given as

$$e^{2\zeta(r)} = \hat{A}_2^2 r^2 \exp \left( - \frac{2k}{\hat{B}_2} \sqrt{\frac{(1 - \frac{R_s}{r})^{-1}}{r^2}} dr \right), \quad (5)$$

where integration constants are  $\hat{A}_2$  and  $\hat{B}_2$ .

The above four conditions are well-known bridge equations that relate both metric functions. The last two conditions are more complicated than the complexity and Karmarkar conditions in deriving the solution of the field equations for compact objects or wormholes. For this purpose, we use the first two conditions in our current problem.

## 2. Background of field equation

The spacetime metric that characterizes a static wormhole with spherical symmetry is written as

$$ds^2 = -e^{2\zeta} dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

where  $R_s(r)$  (shape function) and  $\zeta(r)$  (redshift function) are arbitrary functions that depend on the radial coordinate  $r$ . The massive gravity model in this paper is explained by the following action:

$$\mathcal{H} = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R + m^2 \sum_n^4 k_n \eta_n(u, v)] + \mathcal{H}_{matter}, \quad (7)$$

where  $k_n$  are constant coefficients,  $m$  stands for graviton mass, and  $R$  is the Ricci scalar. It also serves as the free parameter of action. Additionally,  $\eta_n$  is defined as symmetric polynomials of the matrix  $S_\beta^\alpha = \sqrt{g^{\alpha\gamma}} \bar{f}_{\gamma\beta}$  eigenvalues,  $\mathcal{H}_{matter}$  stands for the matter Lagrangian; the reference metric is denoted by  $f_{\alpha\beta}$  and  $g = \det(g_{\alpha\beta})$ . On the other hand,  $\eta_1$ ,  $\eta_2$ ,  $\eta_3$ , and  $\eta_4$  have expressions provided by

$$\eta_1 = [S], \quad (8)$$

$$\eta_2 = [S]^2 - [S^2], \quad (9)$$

$$\eta_3 = [S]^3 - 3[S][S^2] + 2[S^3], \quad (10)$$

$$\eta_4 = [S]^4 - 6[S^2][S^2] + 8[S][S^3] + 3[S^2]^2 - 6[S^4], \quad (11)$$

where  $[S] = [S]_\alpha^\alpha$  and  $[S^n] = (S^n)_\alpha^\alpha$ , and the explicit form of the reference fiducial metric  $f_{\alpha\beta}$  is provided by Vegh; Cai et al. (2015); Adams et al. (2015); Xu et al. (2015)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & J^2 & 0 \\ 0 & 0 & 0 & J^2 \sin^2 \theta \end{pmatrix} \quad (12)$$

where  $J$  is a positive constant.

By changing the action concerning metric  $g_{uv}$ , the updated Einstein field equations can be obtained as follows:

$$G_{\alpha\beta} + \mathfrak{m}^2 \varpi_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (13)$$

where the stress-energy tensor is  $T_{\alpha\beta}$ , the Einstein tensor is  $G_{\alpha\beta}$ , and the massive tensor is  $\varpi_{\alpha\beta}$ . The following uses geometrized units ( $c = G = 1$ ) in its computations. The massive tensor  $\varpi_{\alpha\beta}$  is extracted to form

$$\varpi_{\alpha\beta} = \sum_{n=1}^{d-2} \frac{k_n}{2} \left[ z_n g_{\alpha\beta} + \sum_{y=1}^k (-1)^k \frac{n!}{(n-y)!} z_{n-y} [S_{\alpha\beta}^y] \right], \quad (14)$$

where  $d$  is associated with spacetime's dimensions. As we explore 4-dimensional spacetime,  $d = 4$ . The line element Eq. (6) and reference metric Eq. (12) allow us to calculate the tensor  $S_{\alpha\beta}^y$  is

$$S_{\beta}^{\alpha} = \text{diag} \left( 0, 0, \frac{J}{r}, \frac{J}{r} \right), \quad (15)$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{J^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{J^2}{r^2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{J^3}{r^3} & 0 \\ 0 & 0 & 0 & \frac{J^3}{r^3} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{J^4}{r^4} & 0 \\ 0 & 0 & 0 & \frac{J^4}{r^4} \end{pmatrix},$$

and also

$$[S] = \frac{2J}{r} \quad \& \quad [S^2] = \frac{2J^2}{r^2} \quad (16)$$

$$[S^3] = \frac{2J^3}{r^3} \quad \& \quad [S^4] = \frac{2J^4}{r^4} \quad (17)$$

The only non-zero terms of  $z_n$  are  $z_1$  and  $z_2$ , as we are operating in a 4-dimensional spacetime Hendi et al. (2017).

Consequently,  $z_n$  is found by applying Eqs. (8), (15), and (16)-(17),

$$z_1 = \frac{2J}{r} \quad \& \quad z_2 = \frac{2J^2}{r^2}, \quad (18)$$

$$z_n = 0, \text{ when } n > 2. \quad (19)$$

The massive tensor's elements are found by substituting Eq. (18) into Eq. (14) that is,

$$\varpi_1^1 = \frac{-J(k_2 J + k_1 r)}{r^2}, \quad \& \quad \varpi_3^3 = \frac{-k_1 J}{2r}, \quad (20)$$

$$\varpi_2^2 = \frac{-J(k_2 J + k_1 r)}{r^2}, \quad \& \quad \varpi_4^4 = \frac{-k_1 J}{2r}. \quad (21)$$

Assume for that currently our system's pressure is anisotropic. Therefore, in terms of principal pressure, the energy-momentum tensor  $T_{\alpha\beta}$  can be written as follows:

$$T_{\alpha\beta} = (\rho + P_t) u_{\alpha} u_{\beta} + P_t g_{\alpha\beta} + (P_r - P_t) v_{\alpha} v_{\beta}, \quad (22)$$

with  $u_{\alpha}$  is a timelike unit vector,  $v_{\alpha}$  is a spacelike unit vector orthogonal to the timelike unit vector, such that  $u_{\alpha} u^{\alpha} = -1$  and  $v_{\alpha} v^{\alpha} = 1$ . The Einstein Field Equations (EFEs) in dRGT gravity can be expressed using the line element (6) in an orthonormal reference frame (assuming  $G = c = 1$ ). This yields the following set of equations:

**Table 1**

Shape function criterion for wormhole.

| Shape Function Properties        |                                                                                               |
|----------------------------------|-----------------------------------------------------------------------------------------------|
| $\mathcal{R}_s(r)$               | The throat condition: $0 < \mathcal{R}_s(r) < r$ ; $r > r_0$ and $\mathcal{R}_s(r_0) = r_0$ . |
| $\mathcal{R}'_s(r)$              | Flaring-out condition: $\mathcal{R}'_s(r) < 1$ , for $r_0$ .                                  |
| $\frac{\mathcal{R}_s(r)}{r}$     | Asymptotic flatness condition: For large values of $r$ , it goes to zero.                     |
| $1 - \frac{\mathcal{R}_s(r)}{r}$ | Horizon structure analysis: Incline towards 1 for high $r$ values.                            |

$$8\pi\rho = \frac{\mathcal{R}'_s}{r^2} + \frac{m^2 J(k_2 J + k_1 r)}{r^2}, \quad (23)$$

$$8\pi P_r = 2 \left( 1 - \frac{\mathcal{R}_s}{r} \right) \frac{\zeta'}{r} - \frac{\mathcal{R}_s}{r^3} - \frac{m^2 J(k_2 J + k_1 r)}{r^2}, \quad (24)$$

$$8\pi P_t = \left( 1 - \frac{\mathcal{R}_s}{r} \right) \left[ \zeta'' + \zeta'^2 - \frac{\mathcal{R}'_s r - \mathcal{R}_s}{2r(r - \mathcal{R}_s)} \zeta' - \frac{\mathcal{R}'_s r - \mathcal{R}_s}{2r^2(r - \mathcal{R}_s)} + \frac{\zeta'}{r} \right] - \frac{m^2 k_1 J}{2r}. \quad (25)$$

The equation for conservation can be calculated as  $T_{\alpha;\beta}^j = 0$ , which indicates,

$$\frac{dP_r}{dr} = -(P_r + \rho) \frac{d\zeta}{dr} + \frac{2}{r} (P_t - P_r). \quad (26)$$

### 3. Wormhole models

Taking into consideration the non-zero complexity factor approach and the Karmarkar condition approach, we shall derive new wormhole solutions in the background of dRGT massive gravity in this section. The shape function will be computed using the field equations of dRGT gravity with the nonzero tidal force that is the logarithmic red shift function and should fulfill all the conditions which are discussed in the Table 1.

#### 3.1. Wormhole Model I Null Complexity factor approach

The definition of the complexity factor for a spherically symmetric space-time is Herrera (2018)

$$y_{cf} = \Pi - \frac{1}{2r^3} \int_0^r r^3 \frac{d\rho(r)}{dr} dr, \quad (27)$$

where  $\Pi = P_r - P_t$ . According to the formula,  $y_{cf}$  is zero for an isotropic and homogeneous distribution of matter. The expression clearly shows that  $y_{cf}$  encapsulates all the information about the geometric characteristics of the wormhole Bhattacharya and Nalui (2023). Now our goal is to find a wormhole solution by using a zero-complexity approach. By using Eqs. (23)-(25), we find the value of  $y_{cf}$  is:

$$y_{cf} = - \frac{(r - \mathcal{R}_s) (\zeta'' + (\zeta')^2)}{r} - \frac{J^2 m^2 k_2}{r^2} - \frac{J m^2 k_1}{2r} - \frac{3\mathcal{R}_s}{2r^3} - \frac{\zeta' (\mathcal{R}_s - r\mathcal{R}'_s(r))}{2r^2} + \frac{\mathcal{R}'_s(r)}{2r^2} + \frac{(r - \mathcal{R}_s)\zeta'}{r^2} + \frac{J m^2 r(4Jk_2 + rk_1) - 2r\mathcal{R}'_s(r) + 6\mathcal{R}_s(r)}{4r^3}. \quad (28)$$

Considering the  $y_{cf} = 0$  vanishing of the complexity factor condition, we get

$$\frac{J^2 m^2 k_2}{r^2} + \frac{J m^2 k_1}{2r} + \frac{3\mathcal{R}_s}{2r^3} - \left( \frac{J m^2 r(4Jk_2 + rk_1) - 2r\mathcal{R}'_s(r) + 6\mathcal{R}_s(r)}{4r^3} \right) - \frac{\mathcal{R}'_s(r)}{2r^2} = \frac{(r - \mathcal{R}_s)\zeta'}{r^2} - \frac{(r - \mathcal{R}_s) (\zeta'' + (\zeta')^2)}{r} - \frac{\zeta' (\mathcal{R}_s - r\mathcal{R}'_s(r))}{2r^2}. \quad (29)$$

Here we take red shift function  $\zeta(r) = \log\left(\frac{K}{r}\right)$ . By using the value of  $\zeta$  in above eqn. (29), we obtained the shape function( $\mathcal{R}_s(r)$ ) is

$$\mathcal{R}_s = \frac{1}{10} J m^2 r^2 k_1 + C_1 r^7 + r, \quad (30)$$

where,  $C_1$  is constant of integration. By using throat out condition  $\mathcal{R}_s(r_0) = r_0$ , we obtained the value of the constant of integration, and the final value of the shape function is:

$$\mathcal{R}_s = \frac{1}{10} J m^2 r^2 k_1 + \left( \frac{-J m^2 r_0^2 k_1}{10 r_0^7} \right) r^7 + r. \quad (31)$$

### 3.2. Wormhole Model II Karmarkar condition approach

This section develops the wormhole shape function using the Karmarkar condition, which establishes the wormhole's structure. We take into consideration the static spherically symmetric spacetime metric as in the curvature coordinates.

$$ds^2 = -e^{\varphi(r)} dt^2 + e^{\chi(r)} dr^2 + r^2 d\kappa^2, \quad (32)$$

where  $d\kappa^2 = d\theta^2 + \sin^2\theta d\phi^2$  and  $\varphi(r)$  and  $\chi(r)$  are arbitrary functions of radial coordinate  $r$ . The Riemann curvature components that are not zero are provided by

$$R_{1212} = \frac{e^{\varphi} (2\varphi'' + \chi'^2 - \varphi'\varphi')}{4}, \quad R_{3434} = \frac{r^2 \sin^2\theta (e^{\chi} - 1)}{e^{\chi}}$$

$$R_{1414} = \frac{r \sin^2\theta \varphi' e^{\varphi-\chi}}{2}, \quad R_{2323} = \frac{r \chi'}{2}, \quad R_{1334} = R_{1224} \sin^2\theta.$$

The previously discussed Riemann elements meet the generally accepted Karmarkar constraints as

$$R_{1414} = \frac{R_{1212} R_{3434} + R_{1224} R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0. \quad (33)$$

The space-time that meets the Karmarkar condition is referred to as embedding Class 1. When the values of the Riemann components are replaced in Eq. (33), we obtain

$$\frac{\varphi' \chi'}{1 - e^{\chi}} = \varphi' \chi' - 2\varphi'' - \chi'^2, \quad (34)$$

where  $e^{\chi} \neq 1$ . Consequently, the differential equation above has a solution that is provided in the form of

$$e^{\chi} = 1 + L e^{\varphi} \varphi'^2, \quad (35)$$

where  $L$  is the constant of integration. The shape function is thought to be described by the Morris-Thorne metric, which is given in as eqn. (6). By comparing Eq. (32) and Eq. (6), we get

$$\varphi(r) = 2\zeta(r) \text{ \& } \chi(r) = \log \left[ \frac{r}{r - \mathcal{R}_s(r)} \right]. \quad (36)$$

Now by using the redshift function  $\zeta$  and regarding the equating of the Eqs. (35) and (36), that is to say, the  $\chi(r) = \log \left[ \frac{r}{r - \mathcal{R}_s(r)} \right]$  and  $e^{\chi} =$

$1 + A e^{\varphi} \varphi'^2$ , we obtained the following expression:

$$\left[ \frac{r}{r - \mathcal{R}_s(r)} \right] = 1 + A e^{\varphi} \varphi'^2.$$

The expression for the shape function can be obtained by simplifying the equation above.

$$\mathcal{R}_s(r) = r - \frac{r}{1 + A e^{\varphi} \varphi'^2}. \quad (37)$$

Here,  $\varphi = 2\zeta$  and  $\zeta = \log\left(\frac{K}{r}\right)$  and so,  $\varphi = 2\log\left(\frac{K}{r}\right)$  and  $\varphi' = -\frac{2}{r}$ . Consequently, when we put the values of  $\varphi$  and  $\varphi'$  in Eq. (37), we find

$$\mathcal{R}_s(r) = r - \frac{r^3}{r^2 + 4L \left( \frac{K}{r} \right)^2}. \quad (38)$$

At  $r = r_0$ , we obtain a trivial solution by employing the throat-out condition. So, we add a new free parameter  $\gamma$  for the non-trivial solution that is

$$\mathcal{R}_s(r) = r - \frac{r^3}{r^2 + 4L \left( \frac{K}{r} \right)^2} + \eta, \quad (39)$$

where  $0 < \eta < r_0$ . Now by using throat out condition,  $\mathcal{R}_s(r_0) = r_0$ , we find the value of  $L$  that is,

$$L = \left( \frac{r_0^3}{\eta} - r_0^2 \right) \frac{1}{4 \left( \frac{K}{r_0} \right)^2}. \quad (40)$$

Hence the final obtained shape function is

$$\mathcal{R}_s(r) = r - \frac{r^3}{r^2 + \left( \frac{r_0^3}{\eta} - r_0^2 \right) \frac{1}{\left( \frac{K}{r_0} \right)^2} \left( \frac{K}{r} \right)^2} + \eta. \quad (41)$$

Examining the features of the shape functions covered in Sec. 3, we have plotted in Figs. 1-2, the flaring-out condition ( $-r\mathcal{R}'_s(r) + \mathcal{R}_s$ ) as well as  $\mathcal{R}_s(r)$ ,  $\mathcal{R}'_s$ , and  $\frac{\mathcal{R}_s}{r}$ . Plots of the shape functions can be produced by correctly determining the values for the remaining parameters, which are  $J$ ,  $m$ ,  $k_1$ ,  $k_2$ ,  $K$ ,  $\eta$  and  $r_0$ . For Model I, we vary the values of  $m$  and observe the increasing behavior of the shape function and for Model II, we plot the graphs for different values of  $\eta$  and find the decreasing behavior of the shape function as shown in Figs. 1-2 (upper side left panel). Figs. 1-2 (upper side right panel) illustrates that when  $r > r_0$ , the expression  $\mathcal{R}_s - r$  is negative, indicating that  $\frac{\mathcal{R}_s}{r} < 1$ . This indicates that as  $r$  increases, the function  $\mathcal{R}_s - r$  decreases, supporting the flaring-out requirement for  $r > r_0$  and  $\mathcal{R}'_s < 1$ . Notably, at all radial distances, the requirements  $\mathcal{R}'_s < 1$ ,  $\frac{\mathcal{R}_s}{r} < 1$  for all  $r > r_0$ , and  $\frac{\mathcal{R}_s}{r} \rightarrow 0$  as  $r \rightarrow \infty$  are considerably met. As a result, we can declare with certainty that space-time is asymptotically flat at all radial distances. Here the throat radius is 1.4. The ability to obtain wormhole solutions with reasonable throat sizes is an important step in the theoretical exploration of these exotic spacetime structures. These sizes indicate that the wormhole geometries found in this study are physically feasible because they fall within the range of what has been documented in the literature currently available on wormhole solutions. Theoretically, faster-than-light travel and communication could be possible due to wormholes, which are hypothetical shortcuts in spacetime. Thus, the newly created shape function effectively depicts the wormhole structure.

### 3.3. Embedding diagram

An embedding diagram offers a clear representation of wormhole spacetime and serves as a valuable tool for understanding and visualizing the geometry of both wormholes and spacetime. The embedding surface diagram shows how the tunnel connects two separate points in spacetime, allowing direct travel between the two mouths of the wormhole. It enables us to depict higher-dimensional curved spacetime within a lower-dimensional Euclidean space. This diagram helps us understand the curvature and topology of the wormhole by representing it in a lower-dimensional space, like a two-dimensional plane. By doing so, we can gain insights into the wormhole's shape and characteristics. When constructing an embedding diagram for a wormhole, we assume a spherically symmetric spacetime, meaning the wormhole's geometry is uniform along spherical slices. Specifically, we can focus on an equatorial slice where the angular coordinate  $\theta$  is fixed at  $\frac{\pi}{2}$ . By selecting a constant time slice ( $t = \text{constant}$ ), we can analyze the spatial geometry

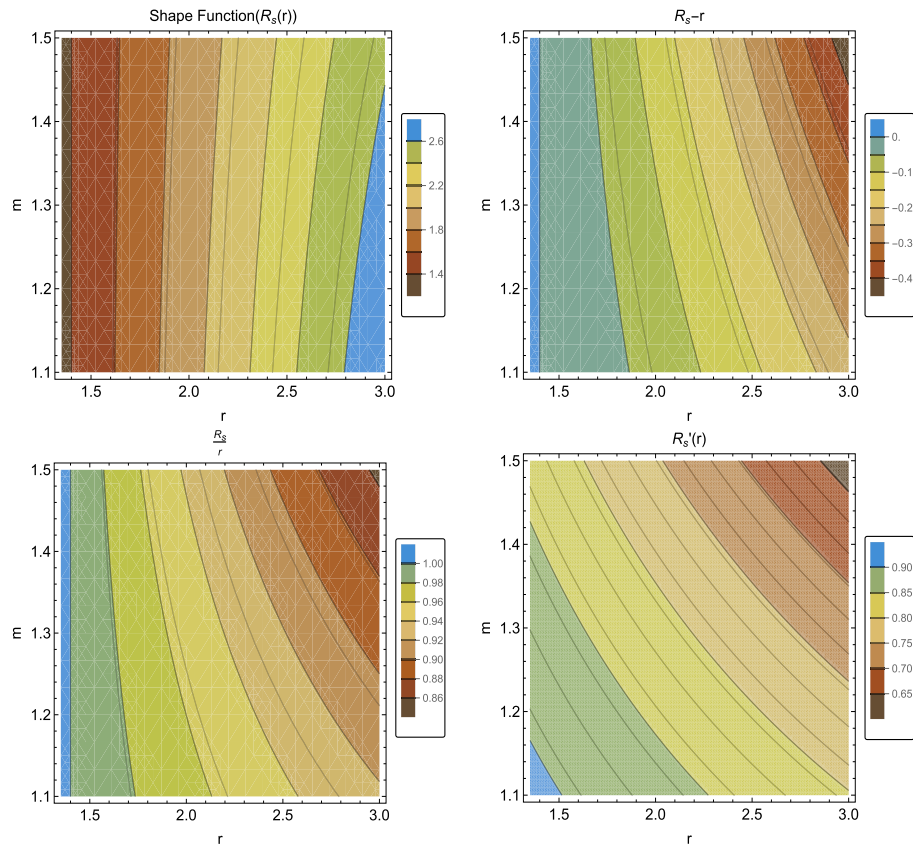


Fig. 1. Diagrammatic representation of Model I for the constraints on the shape function with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$ .

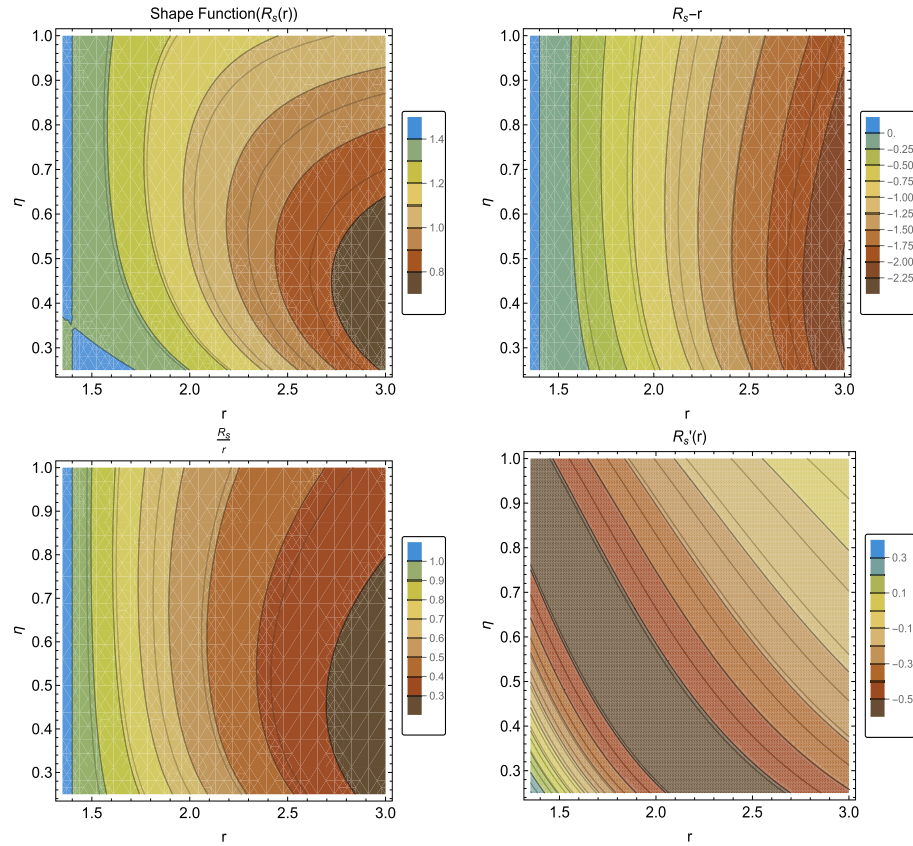
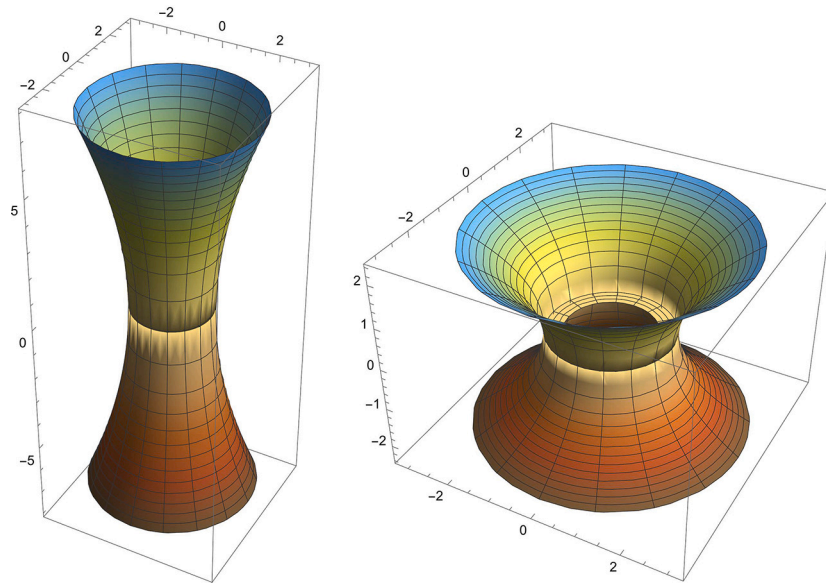


Fig. 2. Diagrammatic representation of Model II for the constraints on the shape function with different values of  $\eta$  and  $r_0 = 1.4$ .



**Fig. 3.** Graphical Representation of Embedding Diagram with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$ ,  $m = 1.5$  and  $k = 0.75$  for Model I (left panel) and Model II (right panel).

of the wormhole without considering the effects of time. Applying these assumptions to Eq. (6), we obtain,

$$ds^2 = \left(1 - \frac{\mathcal{R}_s(r)}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \quad (42)$$

In cylindrical coordinates  $(r, z, \phi)$ , the embedding equation is given as

$$\begin{aligned} ds^2 &= dr^2 + dz^2 + r^2 d\phi^2 \\ &= \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2. \end{aligned} \quad (43)$$

By comparing (42) & (43), we obtain,

$$\frac{dz}{dr} = \pm \left(\frac{r}{\mathcal{R}_s(r)} - 1\right)^{-\frac{1}{2}}. \quad (44)$$

These diagrams serve as a powerful tool for visualizing the curvature and overall shape of a wormhole's space-time. Essentially, embedding diagrams allow us to represent the complex, multi-dimensional structure of space-time in a more relatable, two-dimensional format. By creating these diagrams, we were able to uncover important insights into the geometric properties and characteristics of the wormhole solutions derived from the dRGT massive gravity framework. This framework extends traditional gravity theories and offers a richer set of solutions. The embedding diagrams (See Fig. 3) help us better understand how these solutions behave and interact, revealing details about their stability, potential traversability, and the nature of the space-time curvature involved.

#### 4. Energy condition

In order to explore the possibility of certain cosmic structures, the matter needs to be subjected to a set of constraints called energy conditions. The energy constraints are derived from the Raychaudhuri equation. For the congruences of timelike ( $u^\mu$ ) and null ( $\eta^\mu$ ) geodesics, the temporal development of expansion scalar ( $\theta$ ) is described by the Raychaudhuri equation as Raychaudhuri (1955):

$$\begin{aligned} \frac{d\theta}{d\tau} - \omega_{uv}\omega^{uv} + \sigma_{uv}\sigma^{uv} + \frac{1}{3}\theta^2 + R_{uv}u^\mu u^\nu &= 0 \\ \frac{d\theta}{d\tau} - \omega_{uv}\omega^{uv} + \sigma_{uv}\sigma^{uv} + \frac{1}{2}\theta^2 + R_{uv}\eta^\mu \eta^\nu &= 0 \end{aligned}$$

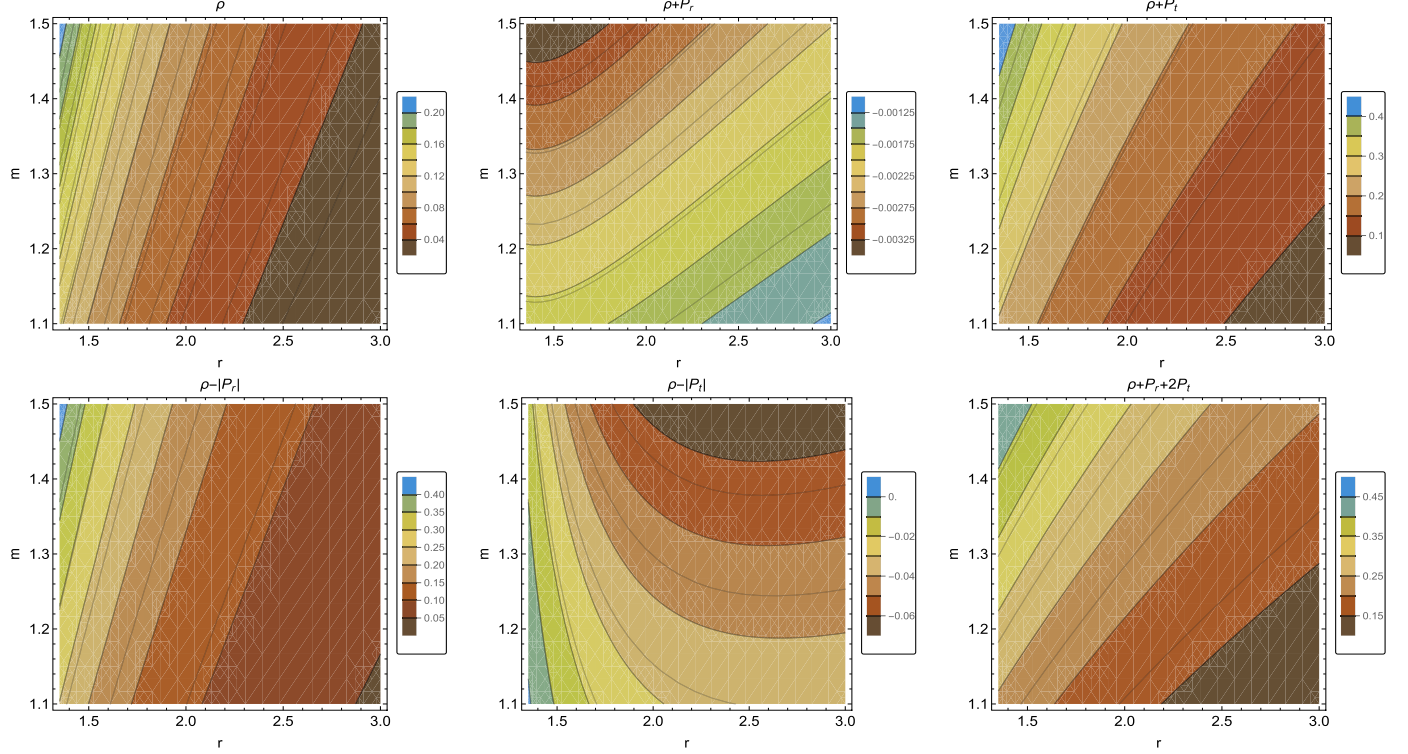
where the shear and rotation associated with the vector field  $u^\mu$  are denoted by  $\sigma^{uv}$  and  $\omega_{uv}$ , respectively. Ignoring the higher-order terms, the Raychaudhuri equations satisfy the following conditions for attractive gravity ( $\theta < 0$ ) as  $R_{uv}u^\mu u^\nu \geq 0$ ,  $R_{uv}\eta^\mu \eta^\nu \geq 0$

The energy conditions are defined as: The weak energy condition (WEC) states that for each timelike observer at any location in space-time, the energy density of any system is positive (in the frame of matter, this translates to  $\rho > 0$  and  $\rho + \mathcal{P} \geq 0$ ), and when the observer functions at the speed of light and has a clearly defined limit known as the null energy condition (NEC) ( $\rho + \mathcal{P} > 0$ ). Since the WEC and NEC are the weakest of all, with the null being even weaker than the weak, their violation indicates that the other energy conditions are also violated Lemos et al. (2003). The main focus of research has been on a technique where an effective energy-momentum tensor that arises within a modified gravitational theory framework violates the energy conditions instead of the matter itself.

Consequently, the analysis of energy conditions establishes the presence of exotic matter as the key component for the throat radius of the wormhole geometry, helping to clarify the nature of the matter involved. Through various ranges of the included model parameter  $m$  for both models and also varies  $\eta$  for Model II, the current research shows that all the energy conditions display positive and negative regions. To analyze the NEC, WEC, Dominant energy condition (DEC), and Strong energy condition (SEC) in terms of  $\rho$ ,  $\mathcal{P}_r$ , and  $\mathcal{P}_t$ , we have created graphs that show the different patterns of each energy condition. The corresponding graphs for both models (Figs. 4–6) provide a visual representation of these energy conditions. The energy density  $\rho$  as a function of  $m$  and  $r$  is plotted graphically, demonstrating that  $\rho$  stays positive (within the region) for the selected range of  $m$  and values of other free parameters. This implies that the system's energy density stays positive, demonstrating the existence of physically plausible matter distributions inside the wormhole shape. The observations for the Model I from Fig. 4 (upper middle and right panel) clearly show that at the wormhole's throat, there is a violation of the  $\rho + \mathcal{P}_r$  but also analyzing the satisfaction of  $\rho + \mathcal{P}_t$ . These conditions indicate the violation of NEC which shows the presence of the exotic matter near the wormhole throat. For the wormhole to remain stable and avoid collapsing, this exotic matter is essential. Additionally, we have noted that the condition DEC at the wormhole's neck has been satisfied in the radial case (see Fig. 4 (lower left)) and violated for the tangential case (see Fig. 4 (lower middle)). Also, the quantity  $\rho + \mathcal{P}_r + 2\mathcal{P}_t$  is positive (Fig. 4 (lower right panel)) hence SEC is satis-

**Table 2**  
Energy conditions for wormhole.

| Energy Conditions         |                                                             | Our Results                                               |
|---------------------------|-------------------------------------------------------------|-----------------------------------------------------------|
| Null energy condition     | $\rho + P_r \geq 0$ & $\rho + P_t \geq 0$ .                 | $\rho + P_r \leq 0$ & $\rho + P_t \geq 0$ .               |
| Weak energy condition     | $\rho \geq 0$ , $\rho + P_r \geq 0$ & $\rho + P_t \geq 0$ . | $\rho \geq 0$ , $\rho + P_r \leq 0$ & $\rho + P_t \geq 0$ |
| Dominant energy condition | $\rho -  P_r  \geq 0$ & $\rho -  P_t  \geq 0$ .             | $\rho -  P_r  \geq 0$ & $\rho -  P_t  \leq 0$ .           |
| Strong energy condition   | $\rho + P_r + 2P_t \geq 0$ .                                | $\rho + P_r + 2P_t \geq 0$ .                              |



**Fig. 4.** Diagrammatic representation of the energy condition for Model I with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$ .

fied. Also, the overall discussion is also discussed in Table 2. For Model II, we check the behavior of energy conditions with a variation of  $m$  and  $\eta$  parameters. From Figs. 5-6, we clearly see that the behaviors of all conditions are the same as in Model I.

In conclusion, energy conditions play a crucial role in shaping the behavior of matter in the cosmos and are necessary to further our comprehension of wormholes.

#### 4.1. Stability

The adiabatic sound velocity can be used to quickly analyze the stability and physical relevance of the solutions which is defined as,

$$v_s^2 = \frac{\partial \langle P \rangle}{\partial \rho}, \quad (45)$$

where

$$\langle P \rangle = \frac{P_r + 2P_t}{3},$$

is symbolized by the mean pressure in the three spatial dimensions.

These essential features remain valid under the following constraints

$$0 \leq v_s^2 < 1.$$

A comprehensive analysis is required to study the potential existence of a repulsive gravitational influence in the wormhole solutions and its impact on the asymptotic structures. Furthermore, by calculating the adiabatic sound velocity and specifically computing the stable region, we examine the configuration's overall stability, as illustrated in Fig. 7.

#### 4.2. Equation of state parameter

Here we evaluate and characterize the parameters of the equation of state (EoS) in the radial directions. EoS creates a connection between the energy density ( $\rho$ ) and the radial pressure ( $P_r$ ) in a physical system, typically a fluid or a material. This equation explains how the pressure of a material changes with its energy density under specific conditions or assumptions. This parameter is defined as

$$\omega = \frac{P_r}{\rho}. \quad (46)$$

The evolution of the Universe could be explained by a barotropic fluid with a positive energy density and an EoS of  $P = \omega\rho$ . The EoS parameter  $\omega$  is crucial in characterizing potential scenarios. Dark energy is the regime where  $-1 < \omega \leq 0$  while the Phantom dark energy, regime is the one where  $\omega \leq -1$  which is known to support traversable wormholes Parsaei and Rastgoo (2019); Kuhfittig (2015). Accelerated expansion of the Universe can be caused by fluid with  $\omega \leq -\frac{1}{3}$  Parsaei and Rastgoo (2019); Kuhfittig (2015). Here we can clearly observe from Fig. 8, the EoS parameter is less than  $-1$ , which shows the region of phantom dark energy.

#### 4.3. Equilibrium condition-TOV equation

The Tolman-Oppenheimer-Volkov (TOV) equation describes the equilibrium structure of spherically symmetric objects, a fundamental equation in astrophysics Tolman (1939). This equation, obtained

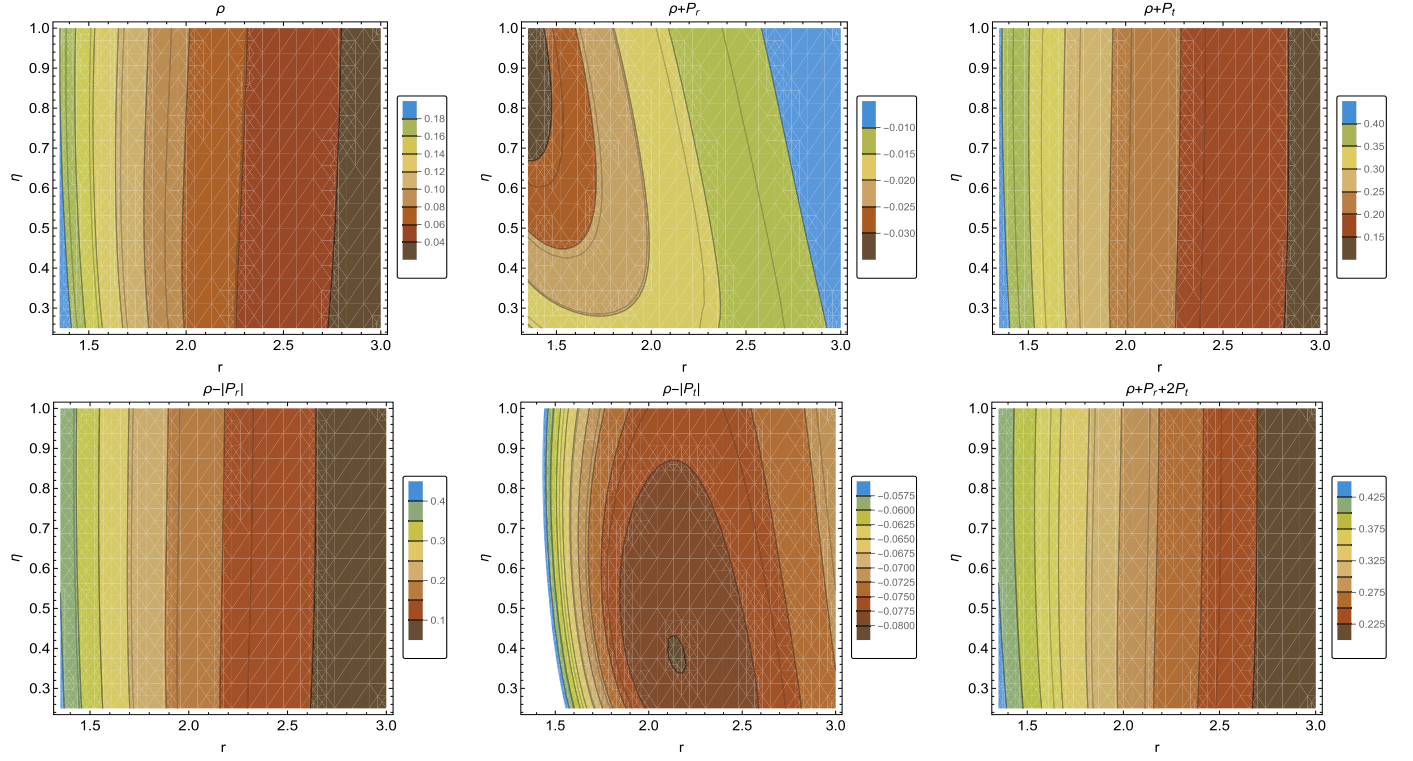


Fig. 5. Diagrammatic representation of the energy condition for Model II with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $m = 1.5$ ,  $r_0 = 1.4$  and different values of  $\eta$ .

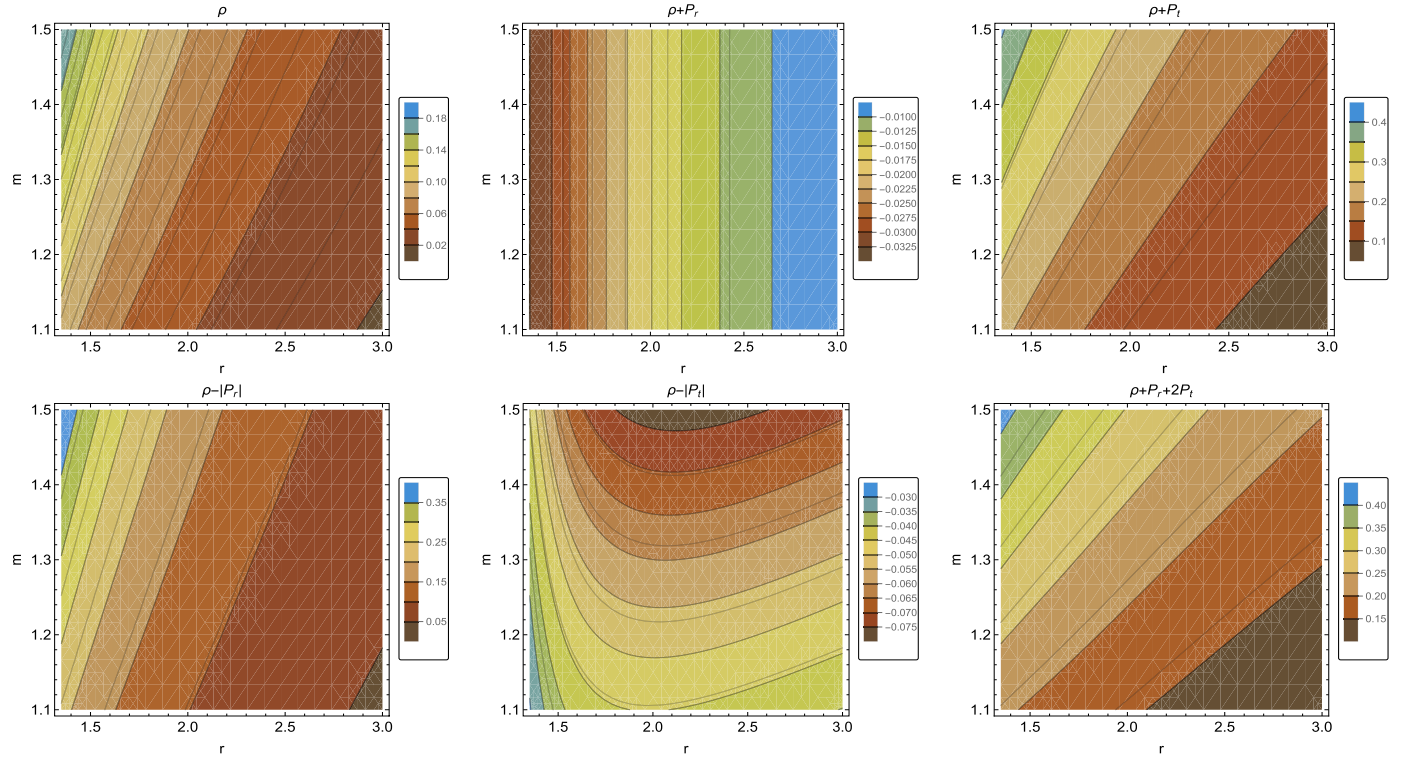
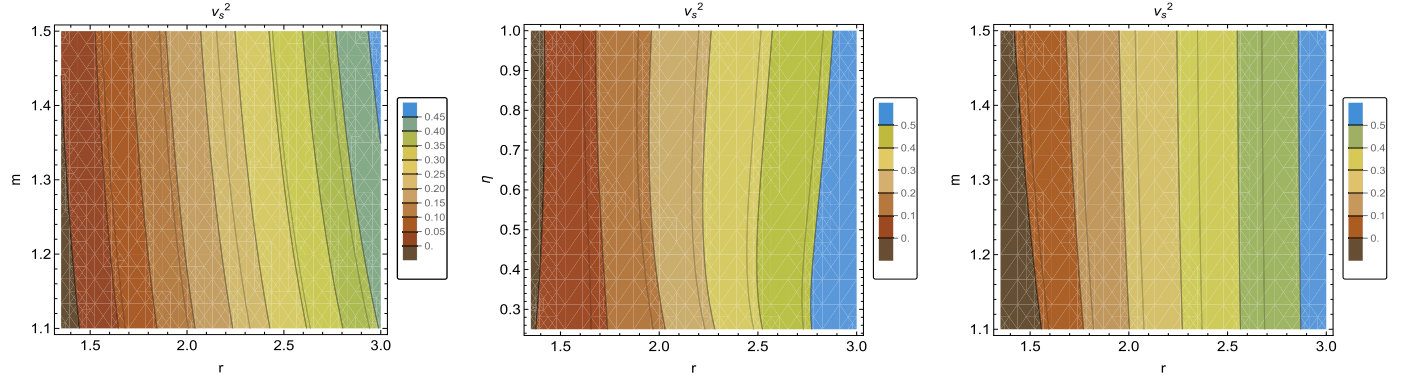
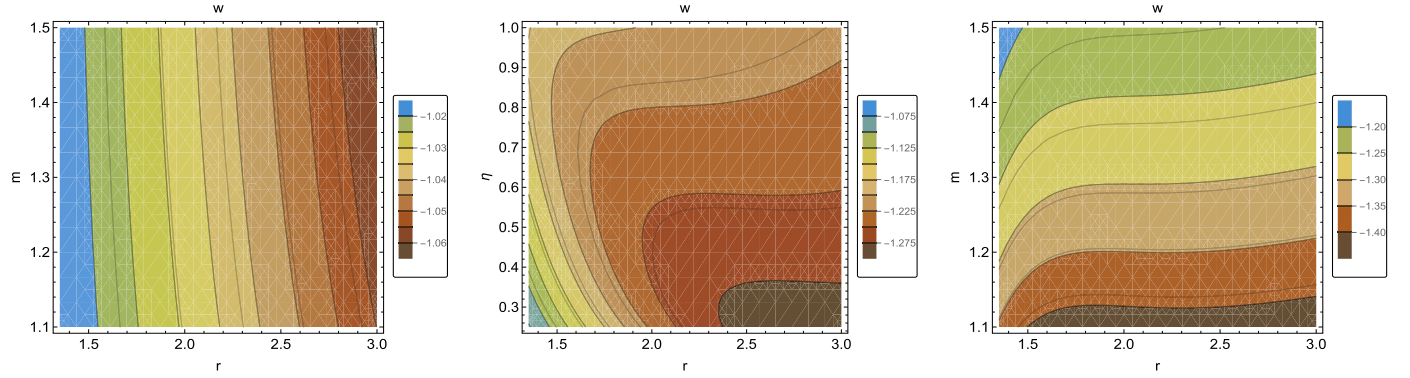


Fig. 6. Diagrammatic representation of the energy condition for Model II with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$ ,  $\eta = 0.75$  and different values of  $m$ .



**Fig. 7.** Diagrammatic representation of the Speed of Sound with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$  for Model I (left panel) and for Model II, varies with different values of  $\eta$  and  $m = 1.5$  (mid panel), different values of  $m$  and  $\eta = 0.75$  (right panel).



**Fig. 8.** Diagrammatic representation of the EoS parameter with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$  for Model I (left panel) and for Model II, varies with different values of  $\eta$  and  $m = 1.5$  (mid panel), different values of  $m$  and  $\eta = 0.75$  (right panel).

from the field equations, describes how the pressure inside a star is connected to its mass and radius. According to the TOV equation, an equilibrium configuration results when the pressure pushing outward balances the gravitational attraction drawing matter inward. In order to comprehend the theoretical prerequisites and limitations for the existence of stable traversable wormholes, it is necessary to investigate the TOV equation within the context of wormhole structures. The feasibility of exotic matter required to support such structures is investigated by researchers using modified gravitational theories and the TOV equation. This has implications for the study of spacetime geometry and theoretical physics beyond astrophysical circumstances. This equation is stated as follows

$$\frac{dP_r}{dr} + \frac{\alpha'(\rho + P_r)}{2} + \frac{2(P_r - P_t)}{r} = 0, \quad (47)$$

where  $\alpha = 2\zeta(r)$  and  $\zeta(r)$  is redshift function. The equilibrium condition for the distribution of matter that makes up the wormhole and is subject to the forces of gravity ( $F_{gf}$ ), hydrostatic pressure ( $F_{hf}$ ), and additional force ( $F_{af}$ ) due to anisotropy is expressed by the Eq.(47). Therefore, the following are the ways in which the gravitational, anisotropic, and hydrostatic forces are defined as Tolman (1939); Oppenheimer and Volkoff (1939)

$$F_{gf} = \frac{-\alpha'(\rho + P_r)}{2},$$

$$F_{af} = \frac{2(P_t - P_r)}{r},$$

$$F_{hf} = \frac{-dP_r}{dr}.$$

For the wormhole solutions to be in equilibrium, it is necessary that  $F_{gf} + F_{af} + F_{hf} = 0$ . We have presented the graphs for all forces for each scenario in Figs. 9, 10, and 11, using the relevant formulas. For all

the situations, it is found that the anisotropic force exhibits positive behavior whereas the hydrostatic and gravitational force exhibits negative behavior. We observed that forces exhibit opposite behaviors. The equilibrium of the solutions may lead one to discover that these forces are the same but opposing ones. Therefore, it is reasonable to say that the wormhole solutions we were able to find for each situation are stable.

## 5. Gravitational lensing effects

This section investigates the connection between traversable wormholes and gravitational lensing effects, emphasizing two approaches: the null complexity factor and the Karmarkar condition. It specifically looks at how the deflection of light is influenced when it passes near wormholes, where the distribution of mass and energy distorts spacetime and affects light propagation. Gravitational lensing, a well-explored phenomenon, occurs when light rays bend around massive entities such as black holes and wormholes. Important work by Virbhadra et al. (1998); Virbhadra and Ellis (2000); Claudel et al. (2001) unveiled the concept of eternal light rings-stable orbits capable of capturing light around certain exotic spacetime configurations. Expanding on this, Bozza Bozza (2002) established an analytical framework for modeling lensing effects in strong gravitational fields, which has been extensively applied in later research to investigate the optical properties of wormholes and other cosmic entities. To delve deeper into the deflection angle of photons along null geodesics, we will present a general spherically symmetric and static line element,

$$ds^2 = -h(r)dt^2 + g(r)dr^2 + f(r)d\Omega^2. \quad (48)$$

Now, we use the geodesic equation to characterize the motion of a freely falling body in relation to the surrounding geometric framework Schutz (2014). This equation provides a link between the momentum

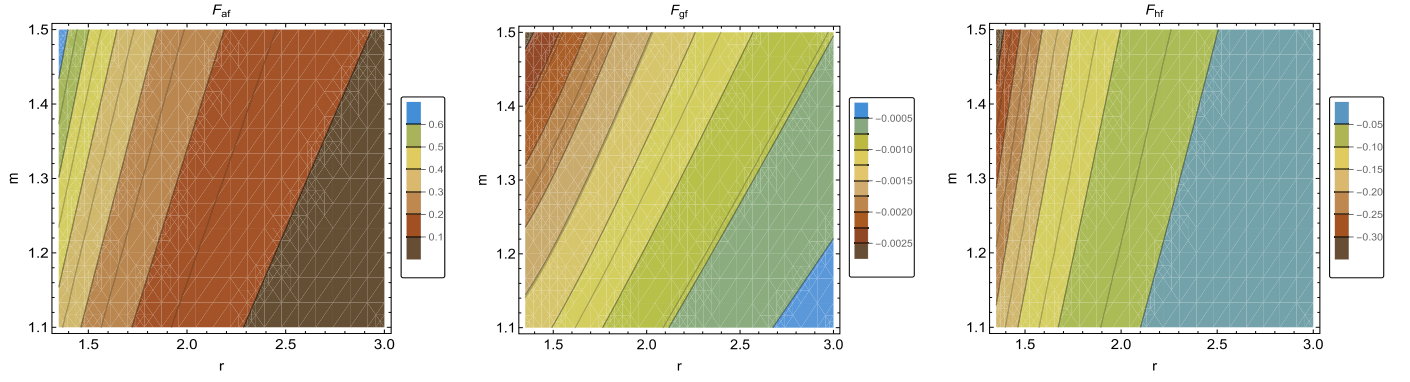


Fig. 9. Diagrammatic representation of the equilibrium condition for Model I with  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$ .

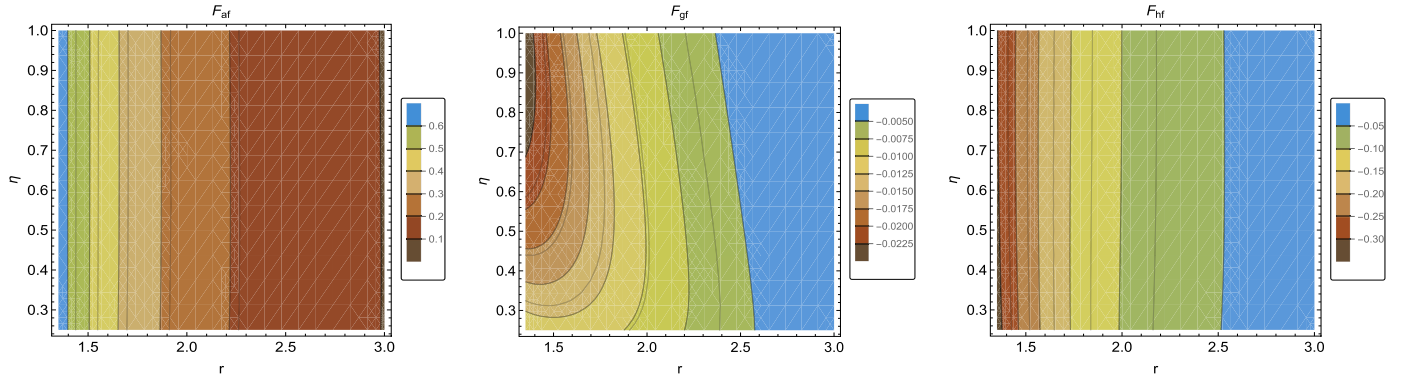


Fig. 10. Diagrammatic representation of the Equilibrium condition for Model II with  $m = 1.5$ ,  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $\eta$ .

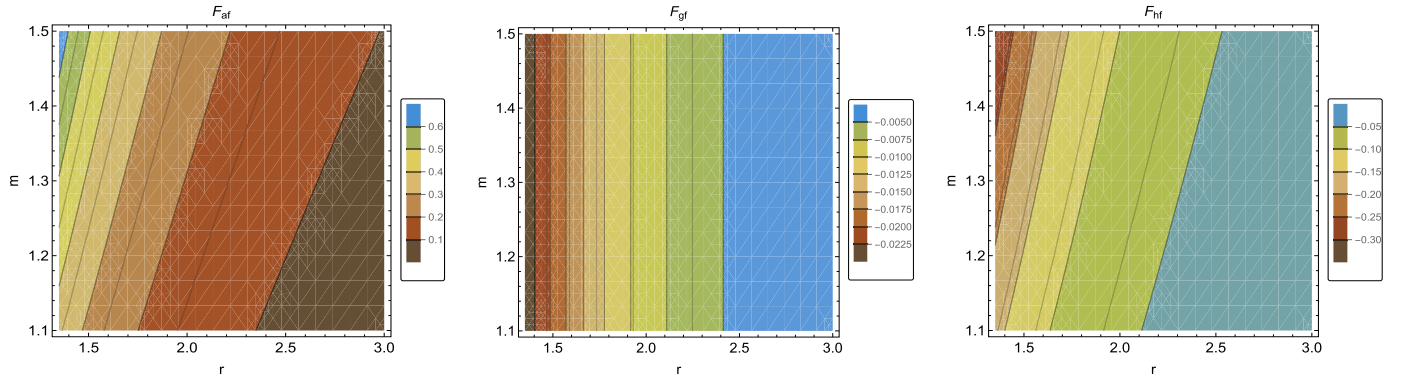


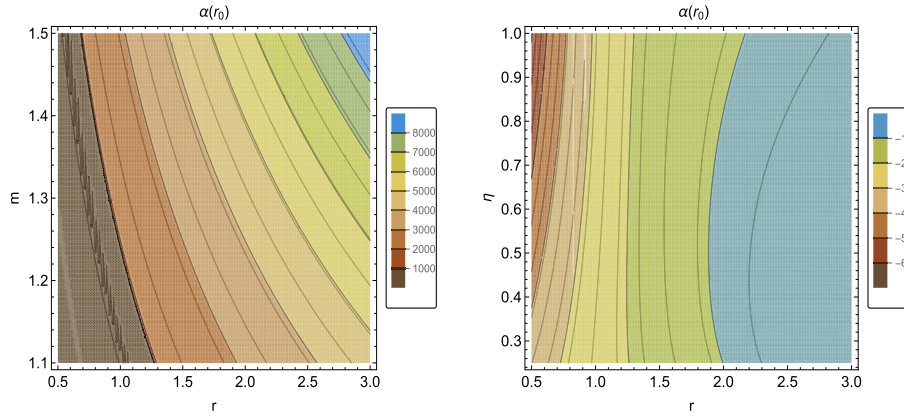
Fig. 11. Diagrammatic representation of the Equilibrium condition for Model II with  $\eta = 0.75$ ,  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$ .

one-forms of the body and the geometric characteristics of spacetime,  $\frac{dp_\beta}{d\lambda} = \frac{1}{2} g_{\alpha\beta} p^\alpha p^\beta$ . In this context,  $\lambda$  denotes the affine parameter. It's important to highlight that if the components of  $g_{\alpha\beta}$  do not depend on a fixed index  $\beta$ , then  $p_\beta$  remains a constant of motion. Focusing on the equatorial plane where  $\theta = \frac{\pi}{2}$ , we observe that all components of  $g_{\alpha\beta}$  become independent of  $t$ ,  $\theta$ , and  $\phi$ . As a result, we can associate the corresponding Killing vector fields  $\delta^\mu_\alpha \partial_\mu$  with  $\alpha$  acting as a cyclic coordinate. By designating the constants of motion as  $p_t$  and  $p_\phi$ , we can further investigate the system's dynamics, where  $p_t = -E$  and  $p_\phi = L$ . Here,  $E$  and  $L$  signify the energy and angular momentum of the photon, respectively. We can then express the geodesic equation in the following manner:  $p_t = i = g^{t\nu} p_\nu = \frac{E}{h(r)}$ , and  $p_\phi = \dot{\phi} = g^{\phi\nu} p_\nu = \frac{L}{f(r)}$ , where the overdot indicates differentiation with respect to the affine parameter  $\lambda$ . Once more, we can straightforwardly derive the radial null geodesic

as follows:  $\dot{r}^2 = \frac{1}{g(r)} \left[ \frac{E^2}{h(r)} - \frac{L^2}{f(r)} \right]$ . In clearer terms, the equation for the photon trajectory can be expressed in relation to the impact parameter  $\mu = L/E$  as,  $\left[ \frac{dr}{d\phi} \right]^2 = \frac{f(r)^2}{\mu^2 g(r)} \left[ \frac{1}{h(r)} - \frac{\mu^2}{f(r)} \right]$ . To analyze the deflection angle of photons, we consider a photon source with a radius  $r_0 > r_a$  that affects the surrounding geometry. For the photons to reach the surface, there must be a solution  $r_0$  that meets the criteria  $r_0 > r_a$  and  $\dot{r}^2 = 0$ . In this scenario,  $r_0$  indicates the distance of closest approach or turning point. Consequently, the impact parameter can be expressed as:

$$\mu = \frac{L}{E} = \pm \sqrt{\frac{f(r_0)}{h(r_0)}}. \quad (49)$$

In the weak gravity limit, where  $\mu \approx \sqrt{f(r_0)}$ , the deflection angle  $\alpha$  of a photon originating from  $\lim_{r \rightarrow \infty} (r, -\frac{\pi}{2} - \frac{\alpha}{2})$ , passing through the turn-



**Fig. 12.** Diagrammatic representation of the deflection angle for both Models I and II with  $K = 1$ ,  $J = 0.915$ ,  $k_1 = 5.12$ ,  $k_2 = -0.298$ ,  $r_0 = 1.4$  and different values of  $m$  and  $\eta$ .

ing point at  $(r_0, 0)$ , and approaching  $\lim_{r \rightarrow \infty} \left( r, \frac{\pi}{2} + \frac{\alpha}{2} \right)$  can be defined explicitly via Eq. (49) (see Virbhadra et al. (1998) for further details). For the chosen metric coefficients in the wormhole geometry, the deflection angle takes the following form

$$\alpha(r_0) = -\pi + 2 \int_{r_0}^{\infty} \frac{e^{\varphi}}{\sqrt{1 - \frac{R_s(r)}{r}} \sqrt{\frac{r^2}{\mu^2} - e^{2\varphi}}} dr. \quad (50)$$

We now apply Eq. (50) to calculate the numerical solution for the deflection angle of photons in dRTG massive gravity. The results for both Models I and II are illustrated in Fig. 12. In Model II, the deflection angle consistently remains negative for all values of  $r_0$ , indicating the presence of repulsive gravity acting on the photons. This negative deflection angle aligns with findings in the literature, where it is shown that such behavior occurs in the corresponding spacetime Panpanich et al. (2019). This finding is significant because it suggests a repulsive gravitational effect in the region surrounding the wormhole. In general, when light passes near a massive object, it is deflected toward that object due to gravitational attraction, resulting in a positive deflection angle. However, the negative deflection angle we found indicates that, rather than being pulled in, light is pushed away from the wormhole. This unusual behavior suggests that the gravitational dynamics near the throat of the wormhole differ fundamentally from the typical gravitational lensing effects we expect to see in standard astrophysical scenarios. Notably, similar conclusions are drawn in analyses involving non-commutative geometry and wormhole solutions with and without massive gravity Dutta et al. (2023); Bhar et al. (2024); Errehymy et al. (2024c). Conversely, Model I exhibits positive deflection angles, reinforcing the idea that the absence of negative deflection angles further supports the observations made in Model II.

## 6. Conclusion

Wormholes are amazing spacetime structures that connect two disparate regions of the same world or Universe. A variety of approaches have been employed in the literature to create feasible wormhole structures. One of them involves formulating the shape functions using different methods, and the other involves looking at how energy constraints behave by taking various shape functions into consideration. The energy conditions can be applied to derive significant physical properties of the object. Within the recently suggested dRTG gravity, this work investigates the occurrence of wormholes in the null zero complexity factor and Karmarkar condition. In order to determine whether or not there are wormhole solutions in this theory, we have constructed a shape function using the complexity method and Karmarkar condition.

The shape function derived using these approaches meets all essential criteria, confirming the existence of a physically viable wormhole geometry (see Figs. 1-2). In Model I, varying  $m$  demonstrates an increasing behavior of the shape function, whereas for Model II, plotting the shape function for different values of  $\eta$  shows a decreasing behavior. For both Model I and Model II, the upper and lower Universes embedded diagrams for the shape functions are depicted in Fig. 3, respectively. Here we consider the non-constant red shift function to check the existence of wormhole solutions.

Our investigation reveals the circumstances in which energy violations take place, suggesting the existence of exotic matter at the throat of the wormhole. The geometric constraints and energy conditions were verified through graphical analysis, showing the possibility of traversable wormholes with the presence of exotic matter for both models. The presence of exotic matter, which results in an NEC violation and enables the wormholes, is one of the most important prerequisites for wormhole construction.

We examined the TOV equation from the perspective of the equilibrium condition for wormhole configuration. Three parts make up the equation: anisotropic, hydrostatic, and gravitational force. As a result, we can conclude that both wormhole solutions are in equilibrium since all forces cancel each other out.

Furthermore, by calculating the adiabatic sound velocity, we observed that it lies between 0 and 1, which shows the stability of the wormhole as illustrated in Fig. 7. We also checked the behavior of the EoS parameter and obtained that  $\omega < -1$  (Fig. 8), which shows the presence of Phantom dark energy.

Moreover, our findings reveal distinct behaviors in the deflection angles of photons in dRTG massive gravity. Model II consistently exhibits negative deflection angles, indicating the presence of repulsive gravity. Conversely, Model I shows positive deflection angles, emphasizing the absence of negative values and further supporting the conclusions drawn in Model II. This contrast highlights the intricate dynamics of gravity and photon behavior across different gravitational approaches.

In conclusion, we can assert that the current method provides a stable and traversable wormhole solution within the arena of dRGT massive gravity theory.

## CRediT authorship contribution statement

**Jitendra Kumar:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Investigation, Formal analysis, Conceptualization. **S.K. Maurya:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Sweeti Kiroriwal:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis,

Conceptualization. **Abdelghani Errehymy**: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Formal analysis, Conceptualization. **Orhan Donmez**: Writing – review & editing, Writing – original draft, Visualization, Formal analysis. **Kairat Myrzakulov**: Writing – review & editing, Writing – original draft, Visualization, Investigation, Formal analysis.

## Declaration of competing interest

None

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## Appendix A

### A.1. The expressions for the solution of model I

$$\rho = \frac{\frac{Jm^2(Jk_2+rk_1)}{r^2} + \frac{\frac{1}{2}Jm^2rk_1+1}{r^2}}{8\pi}, \quad (51)$$

$$P_r = \frac{-\frac{Jm^2(Jk_2+rk_1)}{r^2} - \left(2\left(1 - \frac{\frac{1}{10}Jm^2r^2k_1 - \frac{1}{10}Jm^2r_0^2k_1+r}{r}\right) - \frac{\frac{1}{10}Jm^2r^2k_1 - \frac{1}{10}Jm^2r_0^2k_1+r}{r^3}\right)}{8\pi}, \quad (52)$$

$$P_t = \frac{1 - \frac{\frac{1}{10}Jm^2r^2k_1 - \frac{1}{10}Jm^2r_0^2k_1+r}{r}}{8\pi r^2} - \frac{Jm^2k_1}{2r}. \quad (53)$$

### A.2. The expressions for the solution of model II

$$\rho = \frac{J^2m^2k_2 + Jm^2rk_1 + \frac{r_0^4(\eta-r_0)\left(\eta\left(3r^4+r_0^4\right)-r_0^5\right)}{\left(\eta\left(r^4-r_0^4\right)+r_0^5\right)^2}}{8\pi r^2}, \quad (54)$$

$$P_r = \frac{-r\left(J^2m^2k_2 + Jm^2rk_1 + 1\right) - \frac{\eta r^5}{\eta\left(r^4-r_0^4\right)+r_0^5} + \eta}{8\pi r^3}, \quad (55)$$

$$P_t = \frac{1}{8} \left( -\frac{4Jm^2k_1}{r} - \frac{\eta}{\pi r^3} + \frac{\eta r^2}{\pi \eta r^4 - \pi \eta r_0^4 + \pi r_0^5} \right). \quad (56)$$

## Data availability

No data was used for the research described in the article.

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