



Weakened Gallai–Ramsey number for various graphs of order up to six

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ABSTRACT

Gallai–Ramsey theory examines how edge colorings of complete graphs avoiding rainbow triangles inevitably yield monochromatic subgraphs. In this work, we generalize this framework by introducing the *weakened Gallai–Ramsey number* $gr_t^s(G)$, defined as the smallest integer p such that every Gallai t -coloring of K_p contains a copy of G using at most $s < t$ colors. We determine exact values and bounds for $gr_t^s(G)$ for several graph classes, including complete graphs, books, wheels, and complete bipartite graphs. These results extend classical Gallai–Ramsey theory and provide new insight into the behavior of multicolored subgraphs under constrained colorings.

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1. Introduction

The Gallai–Ramsey theory serves as a compelling extension of classical Ramsey theory, focusing specifically on edge colorings of complete graphs while avoiding the presence of rainbow triangles. In this paper we investigate the structural characteristics of Gallai colorings and their applications in computing certain *weakened Gallai–Ramsey numbers*. These numbers differ from traditional Gallai–Ramsey numbers by ensuring the existence of subgraphs whose edges use a limited number of colors, rather than being monochromatic.

A t -coloring of a graph G is a map $f : E(G) \rightarrow \{1, 2, \dots, t\}$. It is not assumed that such a map is surjective, so a t -coloring of a graph may use fewer than t colors. A *Gallai t -coloring* is a t -coloring that avoids rainbow triangles. In other words, $|\{f(xy), f(yz), f(xz)\}| \leq 2$ for all distinct $x, y, z \in V(G)$. The Ramsey number $r(G_1, G_2, \dots, G_t)$ is the smallest integer p such that every t -coloring of K_p contains a monochromatic subgraph isomorphic to G_i in color i for some $i \in \{1, 2, \dots, t\}$. When $G_1 = G_2 = \dots = G_t = G$, this reduces to $r_t(G)$, the t -color Ramsey number. For an extensive review of known results on Ramsey numbers, see the dynamic survey by Radziszowski [16].

The Gallai–Ramsey number $gr_t(G)$ is the smallest integer p such that any Gallai t -coloring of K_p contains a monochromatic subgraph isomorphic to G . A foundational structural result for Gallai colorings, derived from Gallai's work [7] and reinterpreted in [8], provides the following characterization:

Theorem 1. *Every Gallai-colored complete graph can be constructed by replacing the vertices of a 2-colored complete graph (of order at least 2) with smaller Gallai-colored complete graph.*

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The 2-colored complete graph in this construction is known as the *base graph*, while the substituted complete graphs are referred to as *blocks*. An important consequence of this result is that colors not appearing in the base graph must form disconnected subgraphs, as shown in [8]. Also following theorem is a result from [8].

Theorem 2.

$$gr_k(K_{1,m}) = \begin{cases} \frac{5m-6}{2} & \text{if } m \text{ is even,} \\ \frac{5m-3}{2} & \text{if } m \text{ is odd} \end{cases}$$

Theorem 1 offers a valuable framework for examining Gallai colorings by categorizing them into equivalence classes based on the size of the base graph. This method is frequently utilized to determine upper limits for Gallai–Ramsey numbers. To streamline this approach, it is customary to presume that the base graph possesses minimal order. The subsequent lemma, corresponding to Lemma 3.1 in [13], forbids the existence of base graphs of precisely three vertices:

Lemma 3 ([13]). *If the base graph of a Gallai-colored complete graph is chosen to have minimal order, then its order cannot be 3.*

Another key feature of Gallai colorings is the restriction on the number of colors they can employ, as stated in the following result by Erdős, Simonovits, and Sós [5]:

Theorem 4. *For $n \geq 2$, every Gallai coloring of K_n uses at most $n - 1$ colors.*

This theorem is straightforwardly proven via induction on n . For a comprehensive review of Gallai–Ramsey numbers and related topics, the surveys by Fujita, Magnant, and Ozeki [6], the book by Magnant and Salehi Nowbandegani [13] are recommended. Results on wheel graphs in Gallai-coloring can be seen in [12,18].

The concept of *weakened Gallai–Ramsey numbers* was first introduced by Beam et al. [1], with further developments and results presented in more recent works [2,11]. Denoted by $gr_t^s(G)$, the weakened Gallai–Ramsey number represents the smallest integer p such that every Gallai t -coloring of the complete graph K_p contains a subgraph isomorphic to G , whose edges use at most s colors, where $s < t$. This parameter generalizes both the classical Gallai–Ramsey numbers and the weakened Ramsey numbers, the latter of which were studied in [3,4,9,10]. In particular, when $s = 1$, the weakened Gallai–Ramsey number reduces to the standard Gallai–Ramsey number, i.e., $gr_t^1(G) = gr_t(G)$.

Several foundational results regarding $gr_t^s(G)$ have been established in the literature. Below, we highlight key theorems from recent studies:

Theorem 5 (Beam et al. [1]). *Let $t \geq 3$, $1 \leq s \leq n$, $n' \leq n$, and $n - s = n' - s'$. Then*

$$gr_t^s(n) \leq gr_t^{s'}(n') + n - n'.$$

Lemma 6 (Budden et al. [2]). *For all $t \geq 3$, we have $gr_t^2(K_4) = t + 2$.*

Theorem 7 (Budden et al. [2]). *For $n \geq 4$, every Gallai coloring of K_n contains a C_4 spanned by edges using at most two colors.*

Lemma 8 (Li et al. [11]). *Let G be a Gallai-coloring of a complete graph, $V \subseteq V(G)$, and $v \in V(G) \setminus V$. Then there is at most one color on the edges between v and V that is not used on any edge within V ; that is, $|C(v, V) \setminus C(V)| \leq 1$.*

In this article, the notation $C[xy]$ represents the color of the edge xy , while $C[xG]$ is used to denote the set of colors of all edges connecting the vertex x to the graph G . For example, if G is a subgraph and x is a vertex outside G , $C[xG]$ lists the colors of all edges between x and the vertices in G . These notation simplifies the discussion of edge color patterns and their implications for monochromatic or 2-colored subgraphs.

In this paper, we investigate weakened Gallai–Ramsey numbers for various graph classes. In Section 2, we compute exact values for the complete graphs, specifically $gr_t^s(K_{s+1})$ and $gr_t^s(K_{s+2})$. Next, we turn our attention to book graphs, wheel graphs, and complete bipartite graphs, examining how Gallai colorings influence the existence of these substructures under the constraint of using at most s colors.

A *wheel graph* W_n is formed by connecting a central vertex to all vertices of a cycle C_n , resulting in a graph of order $n + 1$. A *book graph* B_n consists of n triangles sharing a common edge, called the spine, and has order $n + 2$. A *complete bipartite graph* $K_{m,n}$ is composed of two disjoint vertex sets of sizes m and n , with every vertex in one set adjacent to all vertices in the other set, and no edges within the same set, yielding a graph of order $m + n$. These families are particularly well-suited for studying structural guarantees in edge-colored complete graphs under Gallai constraints.

The paper concludes in Section 3 with conjectures and potential directions for further research.

2. Main results

2.1. Results for complete graphs

Theorem 9. *For all $1 \leq s < t$, $gr_t^s(K_{s+1}) = s + 1$.*

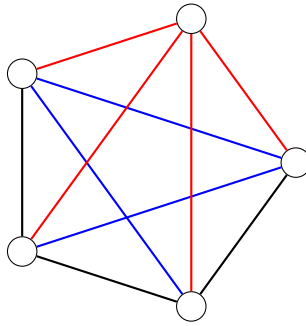


Fig. 1. Critical 3-coloring of K_5 for W_4 .

Proof. By Theorem 4, every Gallai t -coloring of K_{s+1} uses at most s colors. Since at least $s+1$ vertices are needed to form a K_{s+1} , the theorem follows.

In particular, for all $t \geq 3$, $gr_t^3(K_4) = 4$.

Theorem 10. For all $2 \leq s < t$, $gr_t^s(K_{s+2}) = t + 2$.

Proof. First, we construct a Gallai t -coloring of K_{t+1} that avoids a K_{s+2} subgraph spanned by edges using at most s colors. Consider the vertex set $\{v_1, v_2, \dots, v_{t+1}\}$ and if $1 \leq i < j \leq t+1$, then color edge $v_i v_j$ with color $j-1$. Deleting any vertex removes exactly one color from the resulting Gallai t -coloring of K_{t+1} . It follows that the deletion of any $t-s-1$ vertices leaves a K_{s+2} -subgraph spanned by edges using exactly $s+1$ colors. Hence, $gr_t^s(K_{s+2}) \geq t+2$.

To prove the reverse inequality, consider an arbitrary Gallai t -coloring of K_{t+2} . By Lemma 6, there exists a K_4 -subgraph spanned by edges that use at most 2 colors. Denote the vertex set for such a K_4 -subgraph by $\{x_1, x_2, x_3, x_4\}$ and the other vertices by $x_5, x_6, \dots, x_{t+2}$. Note that at most one new color is included in the subgraph induced by $\{x_1, x_2, x_3, x_4, x_5\}$ (see Lemma 8), resulting in a K_5 -subgraph spanned by edges using at most 3 colors. Continuing in this manner, the subgraph induced by $\{x_1, x_2, \dots, x_{s+2}\}$ is a K_{s+2} -subgraph spanned by edges using at most s colors. Thus, $gr_t^s(K_{s+2}) \leq t+2$.

In particular, note that for all $t \geq 4$, $gr_t^3(K_5) = t+2$.

Theorem 11. For any $n \geq 4$, each Gallai coloring of the complete graph K_n contains a subgraph B_2 formed by edges of no more than two distinct colors.

Proof. According to Theorem 7, any Gallai coloring of K_n incorporates a cycle C_4 that uses a maximum of two colors. If this C_4 is monochromatic, our task is complete. However, if the C_4 uses two colors (red and blue), for instance, one of the diagonals must be either red or blue; otherwise, a rainbow triangle would exist, proving the presence of a 2-colored subgraph B_2 .

2.2. Results for wheel graphs

Theorem 12. $gr_t^2(W_4) = t + 3$.

Proof. Let us begin by outlining the construction that establishes the lower bound. Consider the graph $G_3 = G$, where the edges are colored using three colors: 1 (black), 2 (blue), and 3 (red), as illustrated in Fig. 1. This graph does not contain a 2-colored W_4 .

To construct the sequence of graphs $G_4, G_5, \dots, G_t$, we start with the initial graph G_3 . For each i from 4 to t , we add a new vertex to the graph and connect it to every vertex in G_{i-1} using edges of color i . This results in a new graph G_i , which contains one additional vertex compared to G_{i-1} , with all the newly added edges assigned the color i . Repeating this process for all i from 4 to t generates the sequence of graphs $G_4, G_5, \dots, G_t$.

It is important to observe that the graph G_t constructed in this manner does not contain any rainbow K_3 or any 2-colored W_4 . Therefore, we conclude that $gr_t^2(W_4) \geq t+3$.

We will demonstrate that any Gallai t -coloring of K_{t+3} necessarily includes a 2-colored W_4 subgraph. Let H represent the minimal size base graph for Gallai coloring, and let H_i denote the blocks of H . According to the Gallai partition theorem, H is identified as a 2-colored graph. If $|V(H)| \geq 5$, a 2-colored W_4 subgraph can be derived by choosing one vertex from each block. Additionally, according to Lemma 3, it is established that $|V(H)| \neq 3$. Consequently, we must examine the circumstances in which $|V(H)| = 2$ and $|V(H)| = 4$ individually.

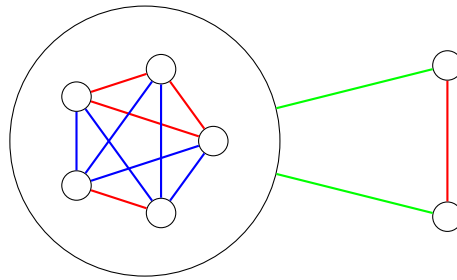


Fig. 2. Critical 3-coloring of K_7 for W_5 .

Case 1: Let us consider the scenario where $|V(H)| = 2$. Assuming the edges connecting the two blocks are red, we can analyze the situation without loss of generality, noting that other colors are confined to the interiors of the blocks.

Subcase 1.1: It is assumed that each block comprises a minimum of two vertices. In accordance with the requirement that at least one block contains a minimum of three vertices, designate three vertices in one block as v_1, v_2 , and v_3 , while assigning two vertices in the other block as v_4 and v_5 . A monochromatic P_3 must exist in the subgraph induced by the vertices v_1, v_2 , and v_3 (denoted as $v_1v_2v_3$). In this scenario, a W_4 structure is established with v_2 designated as the hub, and the cycle C_4 is represented by the sequence $(v_1v_4v_3v_5v_1)$.

Subcase 1.2: Assume that $|H_1| = 1$ and $|H_2| = t + 2$. Let v be the vertex in H_1 , and let $v_1, \dots, v_{t+2}$ be the vertices of H_2 .

We proceed by induction on $t \geq 3$. For the base case $t = 3$, we have $|H_2| = 5$. If H_2 contains no red edge, then the subgraph induced by H_2 is a 2-colored K_5 , and since $W_4 \subset K_5$, a 2-colored W_4 must exist.

Now assume $t \geq 4$ and that the statement holds for all smaller values of t . Then H_2 must contain a red edge (for instance, v_1v_2), since otherwise a 2-colored W_4 would exist by the inductive hypothesis. By the pigeonhole principle, there must be at least two vertices (denote them v_3 and v_4) from the set $\{v_3, v_4, \dots, v_{t+2}\}$ such that the edges of the cycle $C_4(v_3v_1v_4v_2v_3)$ are colored with at most two colors, one of which is red. Therefore, a 2-colored W_4 exists with v as the hub.

Case 2: Assume that $|V(H)| = 4$. Assume, without loss of generality, that the edges linking these blocks are red and black, while other colors are confined to the blocks themselves. If one block (H_1) comprises two vertices, then combined with two vertices from the other blocks (H_2 and H_3), a red/black C_4 is formed. Consequently, a W_4 exists with the hub originating from H_4 .

Theorem 13. $gr_t^2(W_5) = 2(t + 1)$.

Proof. Let us start by explaining the construction that resulted in the lower bound. Consider the graph $G_3 = G$ with edges in color 1 (red), 2 (blue), and 3 (green) as shown in Fig. 2, which does not contain a 2-colored W_5 . To construct $G_4, G_5, \dots, G_t$, we start with the initial graph G_3 . For each value of i from 4 to t , we add a new red edge to the graph and connect it to every vertex in G_{i-1} using edges of color i . This generates a new graph G_i that has two more vertices than G_{i-1} , and all the edges added in this step are colored with the value i . We repeat this process for all values of i from 4 to t to obtain the sequence of graphs $G_4, G_5, \dots, G_t$. It is worth noting that the graph represented by G_t does not contain any rainbow K_3 or any 2-colored W_5 . Hence, $gr_t^2(W_5) \geq 2t + 2$.

We will demonstrate that any Gallai t -coloring of K_{2t+2} necessarily includes a 2-colored W_5 subgraph. Let H represent the minimal size base graph for Gallai coloring, and let H_i denote the blocks of H . According to the Gallai partition theorem, H is identified as a 2-colored graph. If $|V(H)| \geq 6$, a 2-colored W_5 subgraph can be derived by choosing one vertex from each block. Additionally, according to Lemma 3, it is established that $|V(H)| \neq 3$. Consequently, we must examine the circumstances in which $|V(H)| = 2$, $|V(H)| = 4$, and $|V(H)| = 5$ individually.

Case 1: If $|V(H)| = 2$, we can assume that the edges connecting the two blocks are red (designated as color 1), while the other colors are restricted to the blocks themselves.

Subcase 1.1: Assume $|H_1| = 1$ and $|H_2| = 2t + 1$. Let v be the single vertex in H_1 , and let $v_1, v_2, \dots, v_{2t+1}$ represent the vertices in H_2 .

We proceed by induction on $t \geq 3$. For the base case $t = 3$, we have $|H_2| = 7$. If all edges in H_2 avoid red, then H_2 is a 2-colored K_7 . Since $W_5 \subset K_7$, and all colors used are among two, a 2-colored W_5 must exist.

Now assume $t \geq 4$ and that the statement holds for all smaller values of t . To avoid a 2-colored W_5 , there must be red edges within H_2 , because otherwise a 2-colored W_5 would exist by the inductive hypothesis. The block H_2 is essentially a t -colored K_{2t+1} .

We begin by demonstrating that H_2 does not contain any red C_3 . Suppose, for contradiction, $v_1v_2v_3v_1$ is a red C_3 . There can only be t colors from this C_3 to $\{v_4, v_5, \dots, v_{2t+1}\}$. According to Lemma 8, this situation leads to the presence of a 2-colored C_5 (one red and one from the t -color set), creating a 2-colored W_5 with the hub at v , a contradiction.

Next, we demonstrate that H_2 does not contain any red P_3 . Suppose $v_1v_2v_3$ is a red P_3 . There is only t color between v_2 and $\{v_4, v_5, \dots, v_{2t+1}\}$, implying at least two vertices (e.g., v_4, v_5) share the same color with v_2 (let us call it blue). Thus, as per Lemma 8, a red/blue C_5 arises among v_1, v_2, v_3, v_4 , and v_5 , leading once again to a 2-colored W_5 with v as the hub. Finally, we establish that H_2 is devoid of any red $2K_2$. Let us postulate, for argument's sake, that the vertices v_1, v_2, v_3 , and v_4 form a red $2K_2$ (denoted by $C[v_1v_2] = C[v_3v_4] = \text{red}$). As previously demonstrated and consistent with Lemma 8, this induces a monochromatic cycle $C_4(v_1v_3v_2v_4v_1)$, specifically a blue C_4 . Consequently, for any $1 \leq i \leq t$, if $C[v_1v_i] = \text{red/blue}$, a red/blue C_5 is consequently induced using the vertex set $\{v_1, v_3, v_2, v_4, v_i\}$. Edges from the vertices $\{v_5, v_6, \dots, v_{2t+1}\}$ to C_4 are limited to $t - 2$ colors, which implies that there are at least three vertices (for instance, v_5, v_6, v_7) such that $C[v_5C_4] = C[v_6C_4] = C[v_7C_4]$. This results in the induction of a 2-colored C_5 (either red or any of the $t - 2$ available colors) characterized by the cycle $v_5v_3v_6v_2v_1v_5$. Furthermore, a 2-colored W_5 with the hub at vertex v is also induced.

Subcase 1.2: Assume $|H_1| = 2$ and $|H_2| = 2t$. Let x and y denote the vertices in H_1 , while $v_1, \dots, v_{2t}$ represent the vertices in H_2 . Assume the edge xy is black.

We proceed by induction on $t \geq 3$. For the base case $t = 3$, we have $|H_2| = 6$. If either the red or black color is not present in H_2 , then the subgraph induced by H_2 is a 2-colored K_6 . Since $W_5 \subset K_6$, it must contain a 2-colored W_5 .

Now assume $t \geq 4$ and that the statement holds for all smaller values of t . To avoid a 2-colored W_5 , the red and black edges must appear within H_2 ; otherwise, by the inductive hypothesis, a 2-colored W_5 would exist. Assign a red color to edge v_1v_2 and a black color to v_3v_4 . If any red or black edge connects v_1, v_2 and v_3, v_4 , a red or black C_5 is created by x, v_1, v_2, v_3, v_4 , leading to a 2-colored W_5 with hub y .

Should all the edges between v_1, v_2 and v_3, v_4 be monochromatic (apart from red/black), then a C_4 (considered blue) is formed by v_1, v_2, v_3, v_4 . Consequently, a 2-colored wheel W_5 involving hub v_1 and rim $xv_3yv_4v_2x$ emerges.

Now consider the case when red and black edges share a common vertex; for instance, let the edge v_1v_2 be red and v_2v_3 be black. Assume that no other black edge is present in H_2 ; otherwise, the argument reduces to the previous case. In this setting, if we remove the vertex v_2 , the remaining subgraph $H_2 \setminus \{v_2\}$ contains no black edges. Therefore, the graph induced by $H_2 \cup \{x\} \setminus \{v_2\}$ forms a complete graph K_{2t} with at most $t - 1$ colors. By the inductive hypothesis, such a coloring must contain a 2-colored W_5 . Hence, even in this configuration, a 2-colored W_5 necessarily exists.

Again, consider the case when red and black edges share two common vertices; for instance, let the edge v_1v_2 be red, and v_2v_3 and v_1v_3 be black. Assume that no other black edge is present in H_2 ; otherwise, the argument would be similar to the case where H_2 contains two disjoint edges, one red and one black.

In this setting, if we remove the vertex v_3 , the remaining subgraph $H_2 \setminus \{v_3\}$ contains no black edges. Therefore, the graph induced by $H_2 \cup \{x\} \setminus \{v_3\}$ forms a complete graph K_{2t} with at most $t - 1$ colors. By the inductive hypothesis, such a coloring must contain a 2-colored W_5 . Hence, even in this configuration, a 2-colored W_5 necessarily exists.

Hence, in all possible edge colorings within H_2 , a 2-colored W_5 inevitably appears, completing the inductive step.

Subcase 1.3: Assume that each block contains at least three vertices, all the edges between blocks are of color 1 (red) and $|H_1| + |H_2| = 2t + 2$.

Claim 1. Both blocks do not have any red edges.

Proof. - Let us assume a red edge xy and z is any other vertex in H_1 , and a monochromatic P_3 (say abc) in H_2 . Consequently, a 2-colored wheel W_5 involving hub b and rim $xczayx$ emerges.

Claim 2. If $|H_1|$ contains a monochromatic P_3 (e.g., blue), then $|H_2|$ contains no edge of that color; conversely, if $|H_2|$ contains a monochromatic P_3 , then $|H_1|$ contains no edge of that color.

Proof. Similar to the proof of Claim 1.

Claim 3. No block has a monochromatic $K_4 - e$.

Proof. Suppose the graph $K_4 - e$ is induced by the vertices y_1, y_2, y_3, y_4 in block H_2 , where the edges are colored blue. Since $K_4 - e$ must contain a monochromatic $K_{1,3}$, let y_4 serve as the center of this $K_{1,3}$. Now consider block H_1 , which contains at least three vertices. Selecting two vertices from H_1 (denoted x_1 and x_2) along with y_1, y_2, y_3 forms a 2-colored C_5 in red and blue. Since all edges connecting y_4 to the C_5 are either red or blue, we can conclude that a 2-colored W_5 is formed, with y_4 as the hub.

Claim 4. $C[H_1] \cap C[H_2] = \emptyset$

Proof. Let us assume that $x_i \in H_1$ and $y_i \in H_2$, and $C[x_1x_2] = C[y_1y_2] = 2(\text{blue})$. To avoid a rainbow triangle, for any x_i and y_i (where $i \neq 1, 2$), we must have $C[x_ix_1] = C[x_ix_2]$ and $C[y_iy_1] = C[y_iy_2]$, and these colors must not be red or blue, by Claim 1 and Claim 2. Hence, for each i ($i \neq 1, 2$), a monochromatic (not red or blue) path $P_3(x_ix_1x_2)$ is induced in H_1 , and similarly for H_2 . By Claim 2, either x_1x_2 or y_1y_2 is connected to at least three vertices with the same color. (For example, if we take $t = 5$, then $2t + 2 = 12$, and let $|H_1| = 5$, $|H_2| = 7$. Assume red and blue correspond to colors 1

and 2, respectively. Now, if all the edges from x_i ($i \neq 1, 2$) to $\{x_1, x_2\}$ have color 3, then the only possible colors for all the edges from y_i ($i \neq 1, 2$) to $\{y_1, y_2\}$ are colors 4 and 5. Hence, by the pigeonhole principle, $y_1 y_2$ must be connected to three vertices with the same color.)

From the above, consider $C[\{x_1 x_2\}, \{x_3, x_4, x_5, \dots, x_j\}] = 3$ (green), and for all remaining x_k ($k > j$), let $C[\{x_1 x_2\}, x_k] = c_i$, where $c_i \notin \{\text{green, red, blue}\}$.

From Claim 3, there is no green edge among the vertices x_i for $3 \leq i \leq j$. Now, we claim that there is no green edge among the vertices x_k for $k > j$.

To prove this, suppose for contradiction that there exists a green edge, say $x_k x_{k+1}$. Among the vertices x_i for $3 \leq i \leq j$, a monochromatic path P_3 (say, $\{x_3 x_4 x_5\}$) must be induced, and by Claim 3, this path is not green. Without loss of generality, assume that this P_3 is black.

All edges between $\{x_k, x_{k+1}\}$ and $\{x_1, x_2\}$ are monochromatic with color c_i , where $c_i \notin \{\text{red, blue, green}\}$. To avoid a rainbow triangle, all edges from $\{x_k, x_{k+1}\}$ to the vertices of the path P_3 must be either green or c_i .

Thus, a 2-colored subgraph W_5 (green and c_i) is induced with hub x_{k+1} and cycle $x_3 x_k x_2 x_5 x_1 x_3$, leading to a contradiction.

Also, from Claim 2, H_2 do not have any green edge.

Hence both vertex x_1 and x_2 are common in each green edge. Therefor graph $K_{2t+2} - \{x_1, x_2\}$ is a complete graph of order $2t$ with perfect coloring of $t - 1$, so by induction we get a 2-colored W_5 .

We proceed by induction on $t \geq 3$. For the base case $t = 3$, assume that $4 \leq |H_2| \leq 5$ and $3 \leq |H_1| \leq 4$. By Claim 4, H_2 must induce a monochromatic K_4 . Consequently, combining this with any two vertices from H_1 , we obtain a 2-colored $K_6 - e$, which in turn contains a 2-colored subgraph isomorphic to $W_4 \subset K_5 - e$. This establishes the base case for $t = 3$.

Now, assume the result holds for all values $t' < t$; that is, to avoid a 2-colored W_5 , the graph must have at most $2t' + 1$ vertices (by the induction hypothesis: a t' -colored complete graph of order $2t' + 2$ either contains a rainbow triangle or a 2-colored W_5).

Next, consider the case t . By Claim 4, if fewer than t colors (say $k < t$) are used in H_1 , then the remaining $t - k - 1$ colors must appear in H_2 . Therefore, the total number of vertices is:

$$|H_1| + |H_2| = (2k + 1) + (2(t - k - 1) + 1) = 2t < 2t + 2,$$

which contradicts the assumption that the graph avoids a 2-colored W_5 with order $2t + 2$.

Case 2: Suppose that $|V(H)| = 4$. Without loss of generality, assume that the edges connecting the blocks are red and black, while all other colors appear only within the individual blocks.

If any two blocks (say H_1 and H_2) have cardinality at least 2, then we can form a red/black W_5 as follows: take the vertex set $\{H_3, H_2, H_1, H_2, H_1\}$ to induce the rim of a cycle, and use a vertex from H_4 as the hub. This configuration guarantees the existence of a red/black W_5 .

Now consider the case where three blocks (say H_1, H_2, H_3) have cardinality 1, and H_4 has cardinality $2t - 1$.

We proceed by induction on $t \geq 3$.

Base case $t = 3$: Then $|H_4| = 5$. If neither red nor black appears in H_4 , then H_4 induces a monochromatic K_5 , which must contain a 2-colored W_5 with any of other H_i ($1 \leq i \leq 3$).

Base case $t = 4$: Then $|H_4| = 7$. If red and black are both absent in H_4 , then H_4 induces a 2-colored K_7 , which by the inductive hypothesis must contain a 2-colored W_5 . Hence, in either case, a 2-colored W_5 exists.

Now assume $t \geq 5$ and that the result holds for all smaller values. If neither red nor black appears in H_4 , then the inductive hypothesis implies that a 2-colored W_5 must exist in H_4 . Otherwise, suppose H_4 contains a red edge, say $xy \in E(H_4)$. Then, choosing any vertex $v \in H_4$, and the vertices $\{x, y, H_1, H_2\}$, we can form a red/black rim of a 5-cycle. With the vertex in H_3 acting as the hub, this forms a red/black W_5 .

Case 3: Suppose that $|V(B)| = 5$. If one block (say H_1) contains at least two vertices, then a red/blue W_5 with the cycle $H_2 H_1 H_3 H_1 H_4 H_2$ and hub H_5 exists.

Thus, considering all the possibilities, we conclude that a 2-colored W_5 must exist.

2.3. Results for book graphs

Theorem 14. It holds that $gr_3^2(B_3) = 6$.

Proof. The bound $gr_3^2(B_3) \geq 5$ is derived from Fig. 3. To form a 2-colored B_3 , we must find two vertices such that all edges connecting them to the rest of the graph are colored using at most two colors. However, in the coloring presented in the figure, no such pair exists. In fact, there are four vertices in the graph that are each connected to all other vertices using all three colors. This configuration ensures that no pair of vertices can jointly satisfy the edge-color restriction required to form a 2-colored B_3 , thereby proving that K_5 does not guarantee the existence of such a subgraph and justifying the lower bound.

We will demonstrate that any Gallai 3-coloring of K_6 necessarily includes a 2-colored B_3 subgraph. Let H represent the minimal size base graph for Gallai coloring, and H_i denote the blocks of H . According to the Gallai partition theorem, H is identified as a 2-colored graph. If $|V(H)| \geq 5$, a 2-colored B_3 subgraph can be derived by choosing one vertex from each block. Now, according to Lemma 3, it is established that $|V(H)| \neq 3$. Consequently, we must examine the circumstances in which $|V(H)| = 2$ and $|V(H)| = 4$ individually.

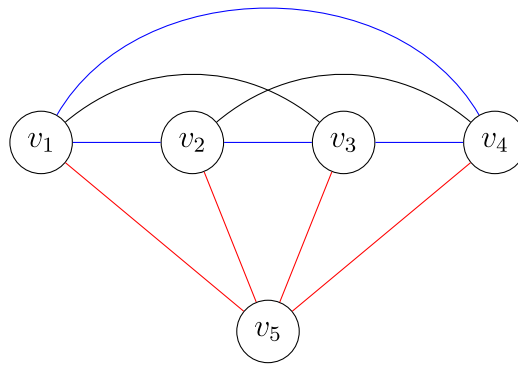


Fig. 3. Critical coloring of K_5 for B_3 .

Case 1: Assume $|V(H)| = 2$. Without loss of generality, mark the edges connecting the two blocks as red, while the other colors, confined to the blocks, as blue and black.

Subcase 1.1: Each block is assumed to contain a minimum of two vertices. At least one block is required to contain a minimum of three vertices. Designate the vertices in one group as v_1, v_2 , and v_3 , while assigning the remaining two vertices as v_4 and v_5 in a separate group. The subgraph induced by vertices v_1, v_2, v_3, v_4 , and v_5 exhibits a 2-colored B_3 , regardless of the color assigned to edge v_4v_5 .

Subcase 1.2: Suppose one block contains only a single vertex (call it u), and the other block includes at least five vertices, denoted by v_1, v_2, v_3, v_4 , and v_5 . The larger block must contain a monochromatic P_3 . Without loss of generality that $v_1v_2v_3$ forms a blue P_3 . If both v_2v_4 and v_2v_5 are black, a B_3 is avoided. The edge v_3v_4 cannot be red; otherwise, it would create a rainbow triangle with v_2, v_3 , and v_4 . Thus, the block produces a blue/black B_3 regardless of the color of the edge v_3v_4 .

Case 2: Let us consider the situation where $|V(H)| = 4$ and the edges that link the four blocks are colored red and black (the remaining colors visible solely inside the blocks). At least one block is required to accommodate two or more vertices, resulting in a B_3 graph with two colors.

Theorem 15. $gr_3^2(B_4) = 7$.

Proof. The inequality $gr_3^2(B_4) \geq 7$ can be deduced from Fig. 4. To obtain a 2-colored B_4 , we require two vertices such that the union of their incident edges to the remaining vertices involves at most two colors. However, the coloring depicted in the figure shows that this is not possible: each vertex (except v) in the graph is connected to other vertices using all three colors. As a result, no pair of vertices can serve as the center of a 2-colored B_4 , since every such pair would collectively be connected to the rest of the graph using all three colors. This demonstrates that K_6 does not necessarily contain a 2-colored B_4 , and thus supports the lower bound of 7.

We will demonstrate that any Gallai 3-coloring of K_7 necessarily includes a 2-colored B_4 subgraph. Let H represent the minimal size base graph for Gallai coloring, and let H_i denote the blocks of H . According to the Gallai partition theorem, H is identified as a 2-colored graph. If $|V(H)| \geq 6$, a 2-colored B_4 subgraph can be derived by choosing one vertex from each block. Now, according to Lemma 3, it is established that $|V(H)| \neq 3$. Consequently, we must examine the circumstances in which $|V(H)| = 2$, $|V(H)| = 4$, and $|V(H)| = 5$ individually.

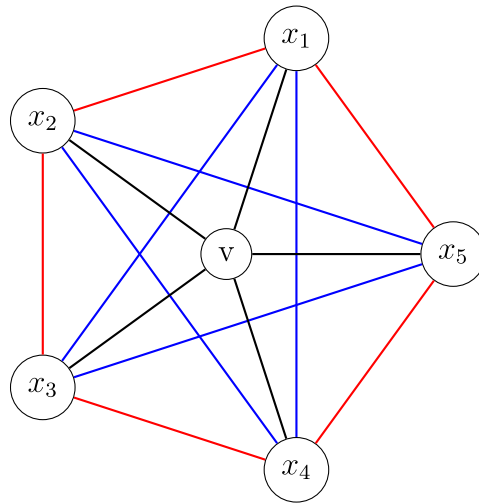
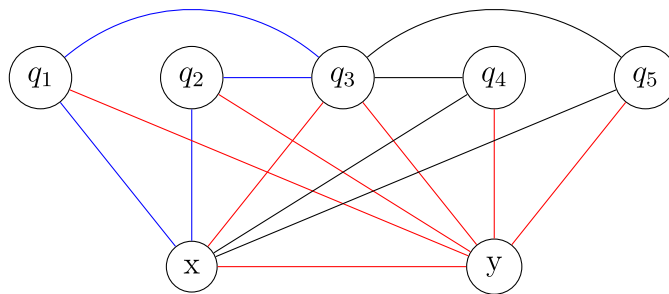
Case 1: Consider $|V(H)| = 2$. Without loss of generality, assume that the edges bridging the two blocks are red, while the remaining two colors (limited to the blocks) are blue and black.

Subcase 1.1: Assume each block has at least two vertices, implying that one block must have at least four vertices. Assign the labels x_1, x_2, x_3 , and x_4 to four vertices within one block, and x_5 and x_6 two vertices within the other. The subgraph formed by x_1, x_2, x_3, x_4, x_5 , and x_6 will then include a 2-colored B_4 , no matter the color of the edge x_5x_6 .

Subcase 1.2: Let us assume there are two blocks: one comprises a single vertex, labeled as y , and the other contains six vertices, specifically q_1, q_2, q_3, q_4, q_5 , and x . According to Theorem 14, a 2-colored B_3 has to be present in this block. Without loss of generality, suppose that the vertices q_1, q_2, q_3, q_4, q_5 form a 2-color B_3 . In scenarios where B_3 is red/blue or red/black, the combination with y results in a 2-colored B_4 . Now consider the situation where B_3 is blue/black, and q_2q_3 is the blue base of the book graph. To avoid a blue/black B_4 , either xq_2 or xq_3 must be red. If both xq_2 and xq_3 are red, the pigeonhole principle dictates that at least two of the edges xq_1, xq_4, xq_5 are either the same color or a mix of red and blue/black. Consequently, a 2-colored B_4 with base xy is formed.

Observation: The following observations lead to the construction of Fig. 5:

- All edges connecting y to the set $\{x, q_1, q_2, q_3, q_4, q_5\}$ are red.
- The edge xq_3 is red, while xq_2 is blue.

Fig. 4. Critical coloring of K_6 for B_4 .Fig. 5. Critical coloring of K_7 for B_4 .

- The edge xq_1 is blue, whereas xq_4 and xq_5 are black. (Precisely two of the edges among xq_1 , xq_4 , and xq_5 must be black; otherwise, a red/blue or red/black B_3 would exist.)
- The edge q_1q_3 is blue; otherwise, a rainbow triangle would exist.
- The edges q_3q_4 and q_3q_5 are black; otherwise, a red/blue B_3 with base q_3 would be formed.

Sub-subcase 1.2.1: If q_2q_4 is blue, then q_2q_1 and q_2q_5 must be black. In this case, the vertices $\{x, q_1, q_2, q_3, q_4, q_5\}$ induce a blue/black B_4 with q_2q_4 as the base, since all edges among $\{q_1, q_4, q_5\}$ are either blue or black.

Sub-subcase 1.2.2: If q_2q_4 is black, then q_4q_1 and q_4q_5 must be blue. In this scenario, the vertices $\{x, q_1, q_2, q_3, q_4, q_5\}$ induce a blue/black B_4 with q_2q_4 as the base, as all edges among $\{q_1, q_4, q_5\}$ are either blue or black.

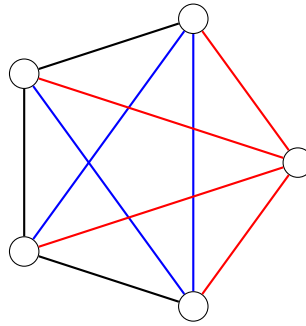
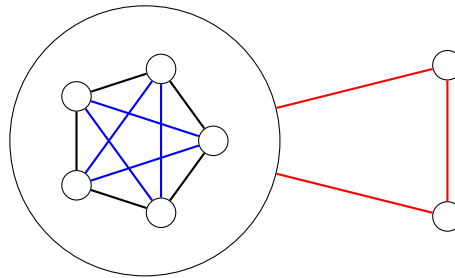
Case 2: Suppose that $|V(H)| = 4$ or $|V(H)| = 5$. Without loss of generality, assume the edges connecting these blocks are red and black, while the remaining colors appear only within the blocks. Since one block must contain at least two vertices, a 2-colored B_4 is guaranteed to exist.

2.4. Results for complete bipartite graphs

Theorem 16. $gr_3^2(K_{2,3}) = 6$.

Proof. $K_{2,3} \subset B_3$ therefore $gr_3^2(K_{2,3}) \leq gr_3^2(B_3) = 6$. For the inequality $gr_3^2(K_{2,3}) \geq 6$ follows from Fig. 6. To guarantee a 2-colored $K_{2,3}$, we must be able to partition the vertex set into two disjoint subsets of size 2 and 3 such that all edges between these subsets use at most two colors. However, the coloring shown in the figure avoids this configuration: in every possible bipartition of the 5 vertices into two sets of 2 and 3, the edges between them span all three colors. Thus, no 2-colored $K_{2,3}$ exists in this coloring. This critical coloring of K_5 demonstrates that it does not suffice to force a 2-colored $K_{2,3}$, thereby establishing the lower bound. Hence $gr_3^2(K_{2,3}) = 6$.

Theorem 17. $gr_3^2(K_{3,3}) = 8$.

Fig. 6. Critical coloring of K_5 for $K_{2,3}$.Fig. 7. Critical coloring of K_7 for $K_{3,3}$.

Proof. The inequality $gr_3^2(K_{3,3}) \geq 8$ follows from Fig. 7. To guarantee a 2-colored $K_{3,3}$, we must be able to partition the vertex set into two disjoint subsets of size 3 such that all edges between these subsets use at most two colors. However, the coloring shown in the figure avoids this configuration: in every possible bipartition of the 7 vertices into two sets of 3, the edges between them span all three colors. Thus, no 2-colored $K_{3,3}$ exists in this coloring. This critical coloring of K_7 demonstrates that it does not suffice to force a 2-colored $K_{3,3}$, thereby establishing the lower bound.

We will demonstrate that any Gallai 3-coloring of K_8 necessarily includes a 2-colored $K_{3,3}$ subgraph. Let H represent the minimal size base graph for Gallai coloring, and H_i denote the blocks of H . According to the Gallai partition theorem, H is identified as a 2-colored graph. If $|V(H)| \geq 6$, a 2-colored $K_{3,3}$ subgraph can be derived by choosing one vertex from each block. Additionally, according to Lemma 3, it is established that $|V(H)| \neq 3$. Consequently, we must examine the circumstances in which $|V(H)| = 2$, $|V(H)| = 4$, and $|V(H)| = 5$ individually.

Case 1: Assume that $|V(H)| = 2$. Without loss of generality, the edges between the two blocks be red, while the other two colors (blue and black) appear only within the blocks. If each block contains at least three vertices, a monochromatic $K_{3,3}$ -subgraph is obtained. Thus, at least one block must contain at most two vertices.

Subcase 1.1: Suppose $|H_1| = 2$ and $|H_2| = 6$. By Theorem 2, a monochromatic star $K_{1,3}$ exists within H_2 . Combining the vertices of this star with those in H_1 produces a 2-colored $K_{3,3}$ -subgraph.

Subcase 1.2: Suppose $|H_1| = 1$ and $|H_2| = 7$. Let v be the vertex in H_1 , and $v_1, \dots, v_7$ the vertices in H_2 . There must be a red edge in H_2 , say v_1v_2 . By the pigeonhole principle, at least three vertices from $\{v_3, v_4, v_5, v_6, v_7\}$ are connected to v_1v_2 using either red/blue or red/black edges. Without loss of generality, let these vertices be v_5, v_6, v_7 . The induced subgraph on $\{v, v_1, v_2, v_5, v_6, v_7\}$ then forms a 2-colored $K_{3,3}$.

Case 2: Now consider $|V(H)| = 4$ or $|V(H)| = 5$. Assume the edges connecting blocks are red and black, while blue appears only within the blocks. If one block has three vertices, these can combine with three vertices from another block to form a $K_{3,3}$. If each block has at most two vertices, we can still combine two blocks with two vertices each and a third block to construct a 2-colored $K_{3,3}$.

3. Conclusion

We propose the following conjecture for the complete graph K_6 . The lower bound for this conjecture is supported by the construction illustrated in Fig. 8. The figure shows a 3-coloring of K_{11} in which every K_6 subgraph contains edges of all three colors. That is, no subset of 6 vertices spans a 2-colored K_6 , as each such subgraph includes at least one edge of each color. This critical coloring demonstrates that K_{11} is insufficient to guarantee the presence of a 2-colored K_6 , thereby establishing the lower bound $gr_3^2(K_6) \geq 12$.

Conjecture 1. $gr_3^2(K_6) = 12$.

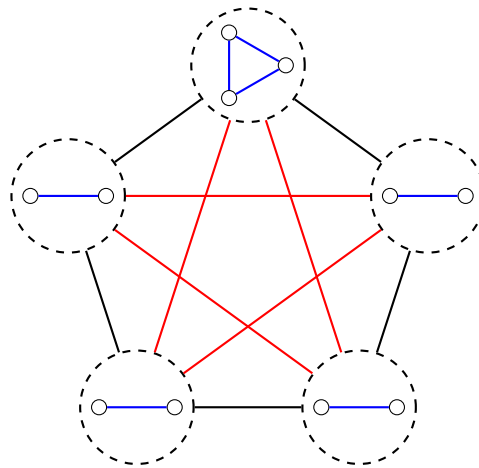


Fig. 8. Critical 3-coloring of K_{11} for K_6 .

This study has determined specific Ramsey values for book graphs, bipartite graphs, wheel graphs, the complete graphs, book, wheel graphs, and complete bipartite graphs providing a foundation for further exploration. Future research could extend these results to broader classes of graphs, such as sun graphs and comb graphs [17], among others. Additionally, investigations into weakened results related to connected Ramsey numbers [14] and restricted-size Ramsey problems [15] may offer deeper insights into structural constraints and generalizations in Ramsey theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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