



On computational analysis via fibonacci wavelet method for investigating some physical problems

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Received: 22 April 2024 / Revised: 11 September 2024 / Accepted: 18 September 2024 /
Published online: 3 October 2024
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Abstract

In this work, we considered wavelet analysis and the application of the Fibonacci wavelet collocation method (FWCM) for solving partial differential equations (PDEs). The proposed technique starts with formulating Fibonacci wavelets using Fibonacci polynomials. Subsequently, the spectral collocation technique is applied to convert the given problem into a system of algebraic equations, which are then solved using the Newton method. Error estimation and convergence analysis of the proposed scheme are also investigated. The effectiveness and precision of the FWCM are demonstrated through a comparative analysis with exact solutions and other existing methods in the literature. The obtained results demonstrate that the proposed technique is an efficient tool for solving PDEs and is also applicable for numerically examining similar types of physical problems.

Keywords Collocation point · Convection diffusion equations · Fibonacci Wavelets · Operational matrices · Klein-Gordon equations

Mathematics Subject Classification 65M99 · 65T60 · 35L10

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1 Introduction

PDEs are mathematical models that represent many physical processes in nature and are used to solve real-world problems such as biological, chemical, and physical problems. A few of the domains that have recently adopted PDEs as their vocabulary include elasticity, general relativity, fluid mechanics [1] sound, heat, and traffic flow [2], thoracic shock remedy for cardiac defibrillation [3], electrophysiology [4], and theoretical and numerical analysis of an optimal control problem related to waste water treatment [5]. A physical phenomenon where particles, energy, or other physical quantities are moved inside a physical system as a result of two processes diffusion and convection is described by the convection-diffusion equations, which combine the diffusion and convection (advection) equations [6]. The PDEs have been solved by a variety of analytical and numerical techniques throughout the last several decades, including the cubic B-spline collocation method. When you add a chemical to water that is flowing when being transported by the river, it diffuses together with a convection term au_x , a diffusion term $b u_{xx}$ also emerges. According to Korkmaz and Dag [7] the equations essentially act as a model for variables including mass, energy, heat, and velocity. They used the cubic B-spline differential quadrature method to solve the said problem. It has been used to represent heat transfer in a draining film and design the movement of water in soils. One of the most significant differential equations in engineering are convection diffusion equations, which have the simplest representation as

$$\frac{\partial \mu}{\partial \tau} + a \frac{\partial \mu}{\partial \varkappa} = b \frac{\partial^2 \mu}{\partial \varkappa^2}, \quad 0 \leq \varkappa \leq 1, \quad \tau \geq 0, \quad (1)$$

with initial conditions(ICs)

$$\mu(\varkappa, 0) = f(\varkappa), \quad 0 \leq \varkappa \leq 1,$$

boundary conditions(BCs)

$$\mu(0, \tau) = 0, \quad \mu(1, \tau) = 0, \quad \tau \geq 0.$$

Where a and b denote the phase speed and the viscosity coefficient, both of which are taken to be positive. The function f that is presented is sufficiently smooth. The behaviour of physical particles like electrons and photons is described by the Klein-Gordon (KG) equations, which are the most crucial mathematical models in plasma physics, nonlinear optics, and quantum field theory. In several areas of physical sciences where the behaviour of wave solitons is significant, the KG equations are crucial [8]. For instance, in the recurrence of starting states and the influence of solitons in collisionless plasma [9]. The quantum amplitude for detecting a point particle at various locations is also described by the KG equations.

$$\frac{\partial^2 \mu}{\partial \tau^2} + a \frac{\partial^2 \mu}{\partial \varkappa^2} + \phi(\mu) = v(\varkappa, \tau), \quad 0 \leq \varkappa \leq 1, \quad \tau \geq 0, \quad (2)$$

with ICs

$$\mu(\varkappa, 0) = h(\varkappa), \quad \frac{\partial \mu(\varkappa, 0)}{\partial \tau} = h_1(\varkappa), \quad 0 \leq \varkappa \leq 1,$$

and BCs

$$\mu(0, \tau) = f_0(\tau), \quad \mu(1, \tau) = f_1(\tau), \quad 0 \leq \tau \leq 1.$$

Where $v(\varkappa, \tau)$, a , $f_0(\tau)$, $f_1(\tau)$, $h(\varkappa)$, and $h_1(\varkappa)$ are functions of dependent or independent constant or variables. To mimic a physical phenomenon, it frequently requires a mathematical model to describe it, and these models are frequently constructed in the form of differential equations employing physics concepts. For a deeper understanding and application of these phenomena, finding solutions to these problems is also necessary. In both physics and mathematics, numerical solutions to PDEs are still essential, and new methods must be created. It is very challenging to find an analytical solution to the PDEs that are used to describe many physical and biological phenomena; numerical approaches play an essential role in the process of finding solutions to these types of problems. The Laplace transform method [10], the finite difference method [11], the generalized finite-difference scheme [12], the finite line method [13], the spectral numerical method [14], the new iteration transform method [15], Adomain decomposition method [16] were used to solve the aforementioned problem. The numerical techniques that are based on wavelets are one of the persuasive alternatives to the existing numerical methods. The wavelet technique outperforms all other approaches, and continuous wavelets outperform discrete wavelets in comparison. The computational study of physical problems has gained popularity in recent years, owing to advances in numerical approaches and computer tools. Wavelet methods have emerged as a popular methodology due to its ability to provide efficient and accurate solutions to a wide range of physics problems. The Fibonacci wavelet technique, in particular, has emerged as an approach to investigate complicated physical processes, including but not limited to problems represented by PDEs like the KG equation and convection-diffusion equations. For some novel work, we refer to [17–19]. Wavelet methods achieved a great success in numerical analysis and approximation theory due to computational simplicity, straightforward methodology, and speedy convergence [20–22]. The most frequently used wavelets for solving differential equations include Legendre wavelets [23], Bernoulli wavelets and Gegenbauer wavelets [24], and Hermite wavelet method [25], Haar wavelets [26]. Fibonacci wavelets are an emerging approach for discovering efficient numerical solutions in the applied and computer sciences [27].

This FWCM may also be used for higher-order equations with minor adjustments. Because of these benefits, Fibonacci wavelets have drawn the attention of numerous researchers. Kumbinarsaiah [28] used the Fibonacci wavelet to solve hyperbolic PDEs. Shah [29] used the Fibonacci wavelet method for non-linear Hunter-Saxton equations and time fractional telegraph equations with Dirichlet boundary conditions. and many more applications of FWCM [30, 31]. Recently, researchers started looking at a useful

method known as the wavelet method to stop these different extreme points based on the results of the prior study and the unique characteristics of PDEs. In this scenario, wavelet approaches are quite simple and decrease processing costs [32]. Wavelet families now include Fibonacci wavelets from Fibonacci polynomials. It has an advantage over other wavelet algorithms. Fibonacci polynomials $P_m(x)$ have fewer coefficients per term compared to Legendre polynomials $L_m(x)$. This characteristic can help minimize computational errors. The coefficients of Fibonacci polynomials can be easily obtained in computer programs using the Mathematica command 'Fibonacci[m, x]'. Fibonacci wavelets are compactly supported and constructed from Fibonacci polynomials over the interval $[0, 1]$. As a direct consequence of this, we investigated two distinct kinds of PDEs using the FWCM. Prior to our work, no one had considered approaching these PDEs using the Fibonacci wavelet. In this study, we will investigate the effectiveness of the Fibonacci wavelet approach in solving physical problems represented by the KG equation and the convection-diffusion equations. We will look at the mathematical framework of the Fibonacci wavelet approach, such as the creation of operational matrices and the use of the FWCM to convert PDEs into algebraic systems. The FWCM determines the approximations accuracy and complexity by use of certain inputs including collocation points, boundary conditions, and polynomials degree. Using local analysis and multiresolution tools, the approach produces effective numerical solutions and sparse representation. Investigation of numerical results to a dynamical system has been done in [33]. It is adaptable to many differential equations, including ODEs and fractional equations such as fractional dynamical model to describe insects [34], COVID-19 model [35, 36], Predator prey model [37] and PDEs [38]. Numerical examples and simulations are presented to illustrate the applicability and accuracy of FWCM in solving KG and convection-diffusion equations, comparing our findings to precise solutions and other considered computational techniques in the literature. Recently, researchers have used different numerical techniques to investigate the solitons solutions for various classes of PDEs, we refer to [39–42]. Because of this, we were motivated to develop the FWCM for solving the KG and convection diffusion equations. In this article several key contributions, it introduces a method using Fibonacci wavelets for solving PDEs, transforms the problem into a solvable system of algebraic equations via spectral collocation, and employs the Newton method for solution. Additionally, the paper provides thorough error estimation and convergence analysis, demonstrating the method's accuracy and efficiency through comparisons with exact solutions and existing methods, establishing the FWCM as a powerful tool for numerical analysis of PDEs and similar physical problems.

Outline of the article: A brief description of the Fibonacci wavelet and function approximation is given in Sect. 2. In Sect. 3, operational matrices of integration are discussed. In Sect. 4, a description of the method for solving the KG and convection diffusion equations and its error analysis are given. In Sect. 5, we considered four test problems to illustrate the efficiency and accuracy of the proposed numerical method. The obtained results show that the proposed technique is an effective tool for solving various other partial differential equations numerically. Section 6 contains a brief conclusion.

2 Fibonacci wavelet

For any $\varkappa \in \mathbb{R}^+$, the recurrence relation defines the Fibonacci polynomials as follows [30]

$$\tilde{P}_{n+2}(\varkappa) = \varkappa \tilde{P}_{n+1}(\varkappa) + \tilde{P}_n(\varkappa), \quad (n \in \mathbb{N}),$$

with ICs

$$\tilde{P}_0(\varkappa) = 0, \quad \tilde{P}_1(\varkappa) = 1.$$

The general formula used to define Fibonacci polynomials is

$$\tilde{P}_{n+1}(\varkappa) = \begin{cases} 1, & n = 0, \\ \varkappa, & n = 1, \\ \varkappa \tilde{P}_n(\varkappa) + \tilde{P}_{n-1}(\varkappa), & n \geq 2. \end{cases}$$

The following is how we define Fibonacci wavelets

$$\Psi_{n,m}(\varkappa) = \begin{cases} \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} \tilde{P}_m(2^{k-1}\varkappa - n + 1), & \frac{n-1}{2^{k-1}} \leq \varkappa < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Where k is resolution level and the translation parameter is n respectively, with $k = 1, 2, 3, \dots$, $n = 1, 2, 3, \dots, 2^{k-1}$ and $\tilde{P}_m$ is the m -degree polynomials. Additionally, the Fibonacci polynomials power-form representation appears as follow

$$\tilde{P}_m(\varkappa) = \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} \varkappa^{m-2i} \quad (m \geq 0). \quad (4)$$

Where $\lfloor \cdot \rfloor$ stands for the well recognized floor function.

$$\begin{aligned} w_m &= \int_0^1 (\tilde{P}(\varkappa))^2 d\varkappa, \quad m = 0, 1, \dots, M-1 \\ &= \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} \binom{m-j}{j} (2(m-i-j)+1)^{-1}. \end{aligned} \quad (5)$$

The Fibonacci polynomial has the following properties

$$\begin{aligned} \int_0^\varkappa \tilde{P}_n(s) ds &= \frac{1}{n+1} [\tilde{P}_{n+1}(\varkappa) + \tilde{P}_{n-1}(\varkappa) - \tilde{P}_{n+1}(0) + \tilde{P}_{n-1}(0)], \\ \int_0^1 \tilde{P}_n(\varkappa) \tilde{P}_m(\varkappa) d\varkappa &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{n-i}{i} \binom{m-j}{j} \frac{1}{n+m-2i-2j+1}. \end{aligned} \quad (6)$$

In particular, if $k = 2$, $M = 4$, the eight basis functions are given by

$$\left. \begin{aligned} \psi_{1,0}(\varkappa) &= \sqrt{2}, \\ \psi_{1,1}(\varkappa) &= 2\sqrt{6}\varkappa, \\ \psi_{1,2}(\varkappa) &= \sqrt{\frac{15}{14}}(1 + 4\varkappa^2), \\ \psi_{1,3}(\varkappa) &= \sqrt{\frac{960}{38}}(2\varkappa^3 + \varkappa), \end{aligned} \right\} 0 \leq \varkappa < \frac{1}{2}. \quad (7)$$

$$\left. \begin{aligned} \psi_{2,0}(\varkappa) &= \sqrt{2}, \\ \psi_{2,1}(\varkappa) &= \sqrt{6}(2\varkappa - 1), \\ \psi_{2,2}(\varkappa) &= \sqrt{\frac{30}{7}}(2\varkappa^2 - 2\varkappa + 1), \\ \psi_{2,3}(\varkappa) &= \sqrt{\frac{480}{304}}(8\varkappa^3 - 12\varkappa^2 + 10\varkappa - 3), \end{aligned} \right\} \frac{1}{2} \leq \varkappa < 1. \quad (8)$$

Any function $f \in L^2[0, 1)$ may be expressed using the Fibonacci wavelets as [30]

$$f(\varkappa) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \Psi_{n,m}(\varkappa). \quad (9)$$

Where

$$C_{n,m} = \langle f, \Psi_{n,m} \rangle = \int_0^1 f(\varkappa) \Psi_{n,m}(\varkappa) d\varkappa,$$

are the coefficients of Fibonacci wavelet. The following is how (9) expressed as a matrix:

$$f(\varkappa) = G^T \Psi(\varkappa), \quad (10)$$

where G is the row vector defined below:

$$G = \left[C_{1,0}, C_{1,1}, \dots, C_{1,M-1}, C_{2,0}, C_{2,1}, \dots, C_{2,M-1}, \dots, C_{2^{k-1},0}, C_{2^{k-1},1}, \dots, C_{2^{k-1},M-1} \right]^T. \quad (11)$$

The matrix $\Psi(\varkappa)$ in (10) is of order $1 \times 2^{k-1}M$ Fibonacci wavelet matrix and is given by

$$\Psi(\varkappa) = \left[\Psi_{1,0}, \Psi_{1,1}, \dots, \Psi_{1,M-1}, \Psi_{2,0}, \Psi_{2,1}, \dots, \Psi_{2,M-1}, \dots, \Psi_{2^{k-1},0}, \right. \\ \left. \Psi_{2^{k-1},1}, \dots, \Psi_{2^{k-1},M-1} \right]^T. \quad (12)$$

Finally, we take into account the collocation points:

$$\varkappa_\ell = \frac{2\ell - 1}{2^k M}, \quad \ell = 1, 2, \dots, 2^{k-1}M. \quad (13)$$

3 Operational matrices of integration

Our objective here is to start formalising the building of operational integration matrices that correlate to Fibonacci wavelets. We emphasise that the methodology is based on Chen's method [18].

$$\int_0^{\varkappa} \Psi_{n,m}(x) dx = P \Psi_{n,m}(\varkappa), \quad (14)$$

where P represents the operational matrix of order $2^{k-1}M \times 2^{k-1}M$. Fibonacci wavelet $\Psi_{n,m}$ can be written as

$$\begin{aligned} \Psi_{n,m}(\varkappa) &= \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} \tilde{P}_m \left(2^{k-1} \varkappa - n + 1 \right) \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}, \\ n &= 1, 2, 3, \dots, 2^{k-1}, \quad m = 0, 1, \dots, M-1, \end{aligned} \quad (15)$$

relation (15) adopts the following format using Fibonacci polynomial power series illustration (4)

$$\Psi_{n,m}(\varkappa) = \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} (2^{k-1} \varkappa - n + 1)^{m-2i} \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}(\varkappa). \quad (16)$$

$$\Psi_{n,m}(\varkappa) = \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m-i}{i} \binom{m-j}{j} 2^{kj-j} \varkappa^j (1-n)^{m-2i-j} \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}(\varkappa). \quad (17)$$

Integrate (17) we obtain

$$\begin{aligned} \int_0^{\varkappa} \Psi_{n,m}(x) dx &= \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m-i}{i} \binom{m-j}{j} 2^{kj-j} \varkappa^j (1-n)^{m-2i-j} F_j(\varkappa), \end{aligned} \quad (18)$$

where

$$\begin{aligned} F_j(\varkappa) &= \int_{n-1/2^{k-1}}^{\varkappa} r^j \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}(\varkappa) dr + \int_{n-1/2^{k-1}}^{n/2^{k-1}} r^j \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}(\varkappa) dr \quad 0 \leq \varkappa \leq m-2i. \end{aligned}$$

The function $F_j(\mathcal{X})$ can also be written as

$$F_j(\mathcal{X}) \simeq \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \alpha_{pq} \Psi_{p,q}(\mathcal{X}). \quad (19)$$

Substituting equation (19) in (18) we have

$$\int_0^{\mathcal{X}} \Psi_{n,m}(x) dx \simeq \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} \theta_{pq}^{n,m} \Psi_{p,q}(\mathcal{X}), \quad (20)$$

where

$$\theta_{pq}^{n,m} = \frac{2^{\frac{k-1}{2}}}{\sqrt{w_n}} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{m-2} \binom{m-i}{i} (2^{kj-j} \mathcal{X})^j (1-n)^{m-2i-j} \alpha_{pq}. \quad (21)$$

As a result, the integration P operating matrix looks like this.

$$P = \begin{pmatrix} \theta_{1,0}^{1,0} & \theta_{1,0}^{1,1} & \cdots & \theta_{1,0}^{1,2^{k-1},M-1} \\ \theta_{1,0}^{1,1} & \theta_{1,1}^{1,1} & \cdots & \theta_{1,1}^{1,2^{k-1},M-1} \\ \vdots & \vdots & \cdots & \vdots \\ \theta_{1,0}^{2^{k-1},M-1} & \theta_{1,1}^{2^{k-1},M-1} & \cdots & \theta_{1,1}^{2^{k-1},M-1} \end{pmatrix}. \quad (22)$$

We consider the case $k = 2, M = 3$, and integrating equation (7) and (8) about the collocations points yields as follows:

$$\begin{aligned} \int_0^{\mathcal{X}} \Psi_{1,0}(r) dr &= \left(0, \frac{\sqrt{3}}{6}, 0, 1/2, 0, 0 \right)^T \Psi_{6 \times 6}(\mathcal{X}), \\ \int_0^{\mathcal{X}} \Psi_{1,1}(r) dr &= \left(-\frac{\sqrt{3}}{4}, 0, \frac{\sqrt{35}}{10}, \frac{\sqrt{3}}{4}, 0, 0 \right)^T \Psi_{6 \times 6}(\mathcal{X}), \\ \int_0^{\mathcal{X}} \Psi_{1,2}(r) dr &= \left(-\frac{29\sqrt{105}}{1680}, \frac{\sqrt{35}}{35}, 1/4, \frac{\sqrt{5}}{21}, 0, 0 \right)^T \Psi_{6 \times 6}(\mathcal{X}), \\ \int_0^{\mathcal{X}} \Psi_{2,0}(r) dr &= \left(0, 0, 0, \frac{\sqrt{3}}{6}, 0 \right)^T \Psi_{6 \times 6}(\mathcal{X}), \\ \int_0^{\mathcal{X}} \Psi_{2,1}(r) dr &= \left(0, 0, 0, \frac{\sqrt{3}}{4}, 0, \frac{\sqrt{35}}{10} \right)^T \Psi_{6 \times 6}(\mathcal{X}), \\ \int_0^{\mathcal{X}} \Psi_{2,2}(r) dr &= \left(0, 0, 0, \frac{29\sqrt{105}}{1680}, \frac{\sqrt{35}}{35}, 1/4 \right)^T \Psi_{6 \times 6}(\mathcal{X}). \end{aligned}$$

Therefore equation (14) takes the form

$$\int_0^{\varkappa} \Psi_{6 \times 1}(x) dx = P_{6 \times 6} \Psi_6(\varkappa).$$

4 Description of method

Consider the KG Eq. (2) as follows

$$\frac{\partial^2 \mu}{\partial \tau^2} + a \frac{\partial^2 \mu}{\partial \varkappa^2} + \phi(\mu) = v(\varkappa, \tau), \quad (23)$$

with ICs

$$\mu(\varkappa, 0) = h(\varkappa), \quad \frac{\partial \mu(\varkappa, 0)}{\partial \tau} = h_1(\varkappa), \quad 0 \leq \varkappa < 1,$$

and BCs

$$\mu(0, \tau) = f_0(\tau), \quad \mu(1, \tau) = f_1(\tau), \quad 0 \leq \tau < 1.$$

Now expanding the highest order derivative we have

$$\ddot{\mu}''(\varkappa, \tau_n) \simeq \sum_{l=0}^{2^{k-1}M} c_l \psi(\varkappa) = G^T \Psi(\varkappa), \quad (24)$$

where c_l , $l = 1, 2, 3, \dots, N$ are unknown Fibonacci coefficient to be calculated and “.” and “.” represent differentiation w.r.t τ and $\varkappa$ respectively.

Integrate Eq. (24) twice w.r.t to τ between limit τ_n to τ we obtain

$$\begin{aligned} \dot{\mu}''(\varkappa, \tau) &= (\tau - \tau_n) \sum_{l=0}^{2^{k-1}M} c_l \psi(\varkappa) + \dot{\mu}''(\varkappa, \tau_n), \\ \mu''(\varkappa, \tau) &= \frac{1}{2} \Delta \tau^2 \sum_{l=0}^{2^{k-1}M} c_l \psi(\varkappa) + \Delta \tau \dot{\mu}''(\varkappa, \tau_n) + \mu''(\varkappa, \tau_n), \end{aligned} \quad (25)$$

where $\Delta \tau = \tau - \tau_n$, integrating (25) w.r.t. $\varkappa$ twice with limits 0 to $\varkappa$

$$\begin{aligned} \mu'(\varkappa, \tau) &= \frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,1}(\varkappa) \\ &+ \Delta \tau [\dot{\mu}'(\varkappa, \tau_n) - \dot{\mu}'(0, \tau_n)] + \mu'(\varkappa, \tau_n) + \mu'(0, \tau) - \mu'(0, \tau_n). \end{aligned} \quad (26)$$

$$\begin{aligned}\mu(\varkappa, \tau) &= \frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \Delta \tau [\dot{\mu}(\varkappa, \tau_n) - \dot{\mu}(0, \tau_n) - \varkappa \dot{\mu}'(0, \tau_n)] + \\ &\quad \mu(\varkappa, \tau_n) - \mu(0, \tau_n) + \mu(0, \tau) - \varkappa [\mu'(0, \tau_n) - \mu'(0, \tau)].\end{aligned}\quad (27)$$

Using BCs, Eq. (27) and put $\varkappa = 1$ can be expressed as

$$\begin{aligned}\mu'(0, \tau) - \mu'(0, \tau_n) &= f_1(\tau) - \frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(1) - \Delta \tau [\dot{f}_1(\tau_n) - \dot{f}_0(\tau_n) \\ &\quad - f'_0(\tau_n)] + f_1(\tau_n) - f_0(\tau) + f_0(\tau_n).\end{aligned}\quad (28)$$

Substituting Eq. (28) in Eq. (27) we obtain

$$\begin{aligned}\mu(\varkappa, \tau) &= \frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \Delta \tau [\dot{\mu}(\varkappa, \tau_n) - \dot{f}_0(\tau_n) - \varkappa \dot{f}'_0(\tau_n)] + \mu(\varkappa, \tau_n) \\ &\quad + f_0(\tau) - f_0(\tau_n) + \varkappa f_1(\tau) \\ &\quad - \frac{\varkappa}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) - \varkappa_l \Delta \tau [\dot{f}_1(\tau_n) - \dot{f}_0(\tau_n) - \dot{f}'_0(\tau_n)] \\ &\quad - \varkappa_l [f_1(\tau_n) + f_0(\tau) - f_0(\tau_n)].\end{aligned}\quad (29)$$

Differentiating (29) twice with respect to τ we obtain

$$\begin{aligned}\dot{\mu}(\varkappa, \tau) &= \Delta \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,1}(\varkappa) + \dot{\mu}(\varkappa, \tau_n) - \dot{f}_0(\tau_n) - \varkappa \dot{f}'_0(\tau_n) + \dot{f}_0(\tau) + \varkappa \dot{f}_1(\tau) \\ &\quad - \varkappa \Delta \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) - \varkappa [\dot{f}_1(\tau_n) - \dot{f}_0(\tau_n) - \dot{f}'_0(\tau_n) + \dot{f}_0(\tau)].\end{aligned}\quad (30)$$

$$\ddot{\mu}(\varkappa, \tau) = \sum_{l=0}^{2^{k-1}M} c_l P_{l,1}(\varkappa) + \ddot{f}_0(\tau) + \varkappa \ddot{f}_1(\tau) - \varkappa \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \varkappa \ddot{f}_0(\tau). \quad (31)$$

After substituting (31) (29) and (25) in (2) we have

$$\begin{aligned}&\sum_{l=1}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \ddot{f}_0(\tau) + \varkappa \ddot{f}_1(\tau) - \varkappa \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \varkappa \ddot{f}_0(\tau) + \\ &\quad a \left(\frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l \psi(\varkappa) + \Delta \tau \dot{\mu}''(\varkappa, \tau_n) + \mu''(\varkappa, \tau_n) \right) + \\ &\quad \phi \left(\frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \Delta \tau (\dot{\mu}(\varkappa, \tau_n) - \dot{f}_0(\tau_n) - \varkappa \dot{f}'_0(\tau_n)) + \right.\end{aligned}$$

$$\begin{aligned}
& \mu(\varkappa, \tau_n) + f_0(\tau) - f_0(\tau_n) + \varkappa f(\tau) \\
& - \varkappa \left(\frac{1}{2} \Delta^2 \tau \sum_{l=0}^{2^{k-1}M} c_l P_{l,2}(\varkappa) + \Delta \tau (\dot{f}_1(\tau_n) - \dot{f}_0(\tau_n) - \dot{f}_1'(\tau_n)) + f_1(\tau_n) + f_0(\tau) - f_0(\tau_n) \right) \\
& = v(\varkappa, \tau).
\end{aligned} \tag{32}$$

This is a system of algebraic equations, solving this system of equations using Newton methods we find the Fibonacci wavelet coefficient c_l after substituting these coefficient in (29) we get the approximate solution.

Similarly for convection diffusion equation, we expand the term containing highest derivative in term of fibonacci wavelets as

$$\dot{\mu}''(\varkappa, \tau_n) = \sum_{l=0}^{2^{k-1}M} c_l \psi(\varkappa) = G^T \Psi(\varkappa), \tag{33}$$

where c_l , $l = 1, 2, 3, \dots, N$ is the Fibonacci coefficient to be calculated and “.” and “.” represent differentiation w.r.t τ and $\varkappa$ respectively.

$$\dot{\mu}(\varkappa, \tau) + a(\mu'(\varkappa, \tau)) = b(\mu''(\varkappa, \tau)). \tag{34}$$

This is the algebraic equation of (1). By integrating the Eq. (33), we find the subsequent derivatives the same as above, and then substituting in (34), we find the Fibonacci coefficient and approximate solution.

4.1 Error estimation and numerical results

In this subsection, we discuss the estimation of error and convergence analysis of Fibonacci wavelets method.

Theorem 4.1 [27] Suppose $\mu \in C^M[0, 1)$ and $Y_M = \text{span}\{\psi_0(\zeta), \psi_1(\zeta), \dots, \psi_{M-1}(\zeta)\}$. If $\mu_M(\zeta) = A^T \hat{F}(\zeta)$ is the best approximation of $\mu(\zeta)$ out of Y_M on the interval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ the error bound of the approximate solution $\mu^*(\zeta)$ by using Fibonacci wavelet on interval $[0, 1)$ would be obtained in the following form:

$$\|e(\zeta)\|_2 = \|\mu - \mu^*\|_2 \leq \frac{R}{M! \sqrt{2M+1}},$$

here $R = \max_{\varkappa \in [0, 1)} |\mu^M(\varkappa)|$

Theorem 4.2 [27] If a continuous function $f(\zeta)$ be a square integrable function defined on $[0, 1)$ which is bounded by some constant $\tilde{M}$ i.e., $|f(\zeta)| \leq \tilde{M}$, then the function $f(\zeta)$ can be expanded as the sum of Fibonacci wavelet and the series converges to $f(\zeta)$ uniformly, i.e.,

$$f(\zeta) = \sum_{m=1}^{2^{k-1}} \sum_{n=0}^{M-1} g_{m,n} \psi_{m,n}(\zeta),$$

where

$$g_{m,n} = \langle f(\zeta), \psi_{m,n}(\zeta) \rangle.$$

The precision of the Fibonacci wavelets collocation method is measured by the absolute error L_2 and maximum absolute errors L_∞ by using the formulas given as

$$\begin{aligned} L_2 &= \|\mu(\varkappa, \tau) - \mu_{n,m}(\varkappa, \tau)\|, \\ L_\infty &= \max |\mu(\varkappa, \tau) - \mu_{n,m}(\varkappa, \tau)|, \end{aligned}$$

where $\mu(\varkappa, \tau)$ and $\mu_{n,m}(\varkappa, \tau)$ are the exact and approximate solutions respectively. The whole computational work is done on (Matlab–R2022a).

5 Illustrative examples

Here, we present some examples to verify our results.

Example 1 Assume that $a = 0.1$ and $b = 0.01$ in (1) as follows:

$$\frac{\partial \mu}{\partial \tau} + 0.1 \frac{\partial \mu}{\partial \varkappa} = 0.01 \frac{\partial \mu^2}{\partial \varkappa^2}, \quad 0 < \varkappa < 1, \quad \tau \geq 0, \quad (35)$$

with ICs

$$\mu(\varkappa, 0) = e^{5\varkappa} \sin(\pi \varkappa), \quad 0 \leq \varkappa \leq 1,$$

and BCs

$$\mu(0, \tau) = 0, \quad \mu(1, \tau) = 0, \quad \tau \geq 0.$$

The analytic solution is

$$\mu(\varkappa, \tau) = e^{5\varkappa - (0.25 - 0.01\pi^2)\tau} \sin(\pi \varkappa).$$

Fig. 1 shows the comparison of exact and approximate solutions in two and three dimensions. Also, the behaviour of solutions has been shown for $\tau = 0.1, 0.3, 0.5$. In Table 1, we compare the maximum absolute error L_∞ for various values of time τ , enabling one to analyze the suggested approach effectiveness and accuracy. Clearly, the Fibonacci wavelet findings are more effective than the approach in [32].

Example 2 Consider the case when $a = 0.22$, $b = 0.5$,

$$\frac{\partial \mu}{\partial \tau} + 0.22 \frac{\partial \mu}{\partial \varkappa} = 0.5 \frac{\partial \mu^2}{\partial \varkappa^2}, \quad 0 < \varkappa < 1, \quad \tau \geq 0, \quad (36)$$

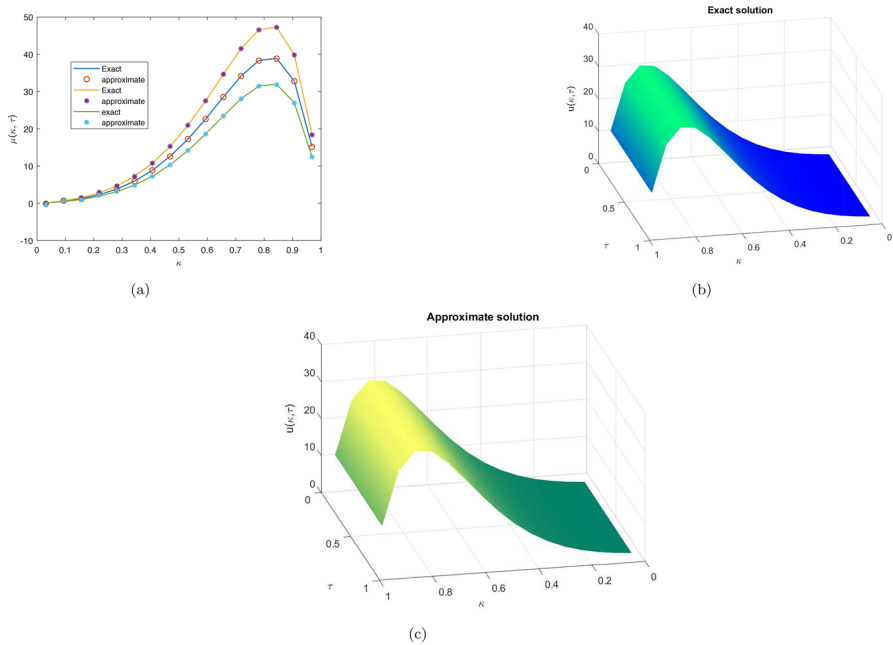


Fig. 1 **a** Comparison of Exact and approximate solution at $\tau = 0.01, 0.02, 0.03$ **b** Exact solution **c** Approximate solution

Table 1 Absolute error(L_∞) are compared at different value of τ of Example 1

($\varkappa$)	FWCM $\tau = 0.1$	HWCM [32] L_∞	FWCM $\tau = 0.3$
0.1	6.2006e-05	1.0E-006	1.6434e-06
0.2	3.6213e-05	2.2E-006	3.3336e-06
0.3	7.2013e-06	3.3E-006	1.1053e-05
0.4	5.3254e-06	3.5E-006	8.2653e-07
0.5	2.1020e-06	5.8E-006	2.7551e-06
0.6	1.6244e-06	2.1E-004	2.0212e-06
0.7	7.0578e-07	2.3E-004	2.7414e-05

with ICs

$$\mu(\varkappa, 0) = e^{0.22\varkappa} \sin(\pi\varkappa), \quad 0 \leq \varkappa \leq 1,$$

and BCs

$$\mu(0, \tau) = 0, \quad \mu(1, \tau) = 0, \quad \tau \geq 0.$$

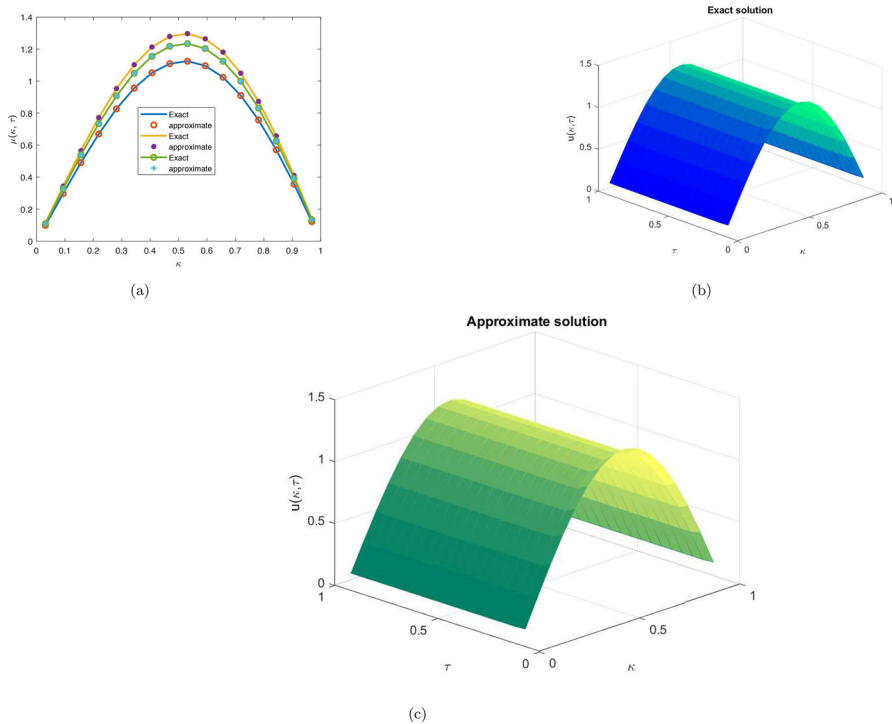


Fig. 2 **a** Exact and Approximate solutions are compared at $\tau = 0.01, 0.03, 0.05$ **b** Exact solution in 3D **c** Approximate solution in 3D

Table 2 For different value of τ the absolute error are compared for Example 2

(x)	FWCM $\tau = 0.01$	HWCM [32] L_∞	FWCM $\tau = 0.03$
0.1	9.5006e-06	7.0E-007	5.2432e-06
0.2	2.6463e-06	2.3E-006	8.6101e-06
0.3	6.7253e-07	4.5E-006	6.2513e-07
0.4	3.6211e-07	7.2E-006	8.0321e-08
0.5	2.0602e-08	1.0E-005	6.7551e-08
0.6	1.6214e-08	1.3E-005	2.0362e-06
0.7	7.0633e-07	2.3E-005	5.7514e-05

The analytic solution is

$$\mu(x, \tau) = e^{0.22x - (0.0242 - 0.05\pi^2)\tau} \sin(\pi x).$$

Fig. 2 shows a comparison of exact and approximate solutions in two and three dimensions. Also, the behaviour of solutions has been shown for $\tau = 0.01, 0.03, 0.05$. In Table 2, we compare the maximum absolute error L_∞ for various values of time τ , enabling one to evaluate the effectiveness and correctness of the proposed approach.

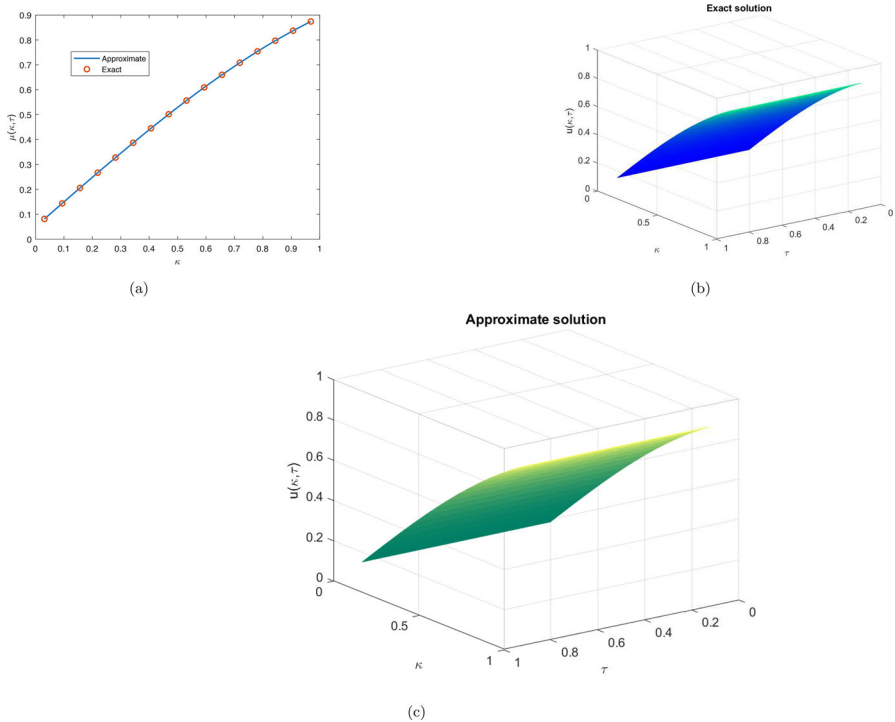


Fig. 3 **a** Comparison of proposed method with exact solutions at $\tau = 0.01$ **b** Exact solution **c** Approximate solution

Obviously, the results obtained with FWCM are more efficient than the method in [32].

Example 3 Consider the KG Eq. (2), we have

$$\frac{\partial^2 \mu}{\partial \tau^2} - \frac{\partial^2 \mu}{\partial \kappa^2} + \mu = 2 \sin(\kappa), \quad (37)$$

with ICs

$$\mu(\kappa, 0) = \sin(\kappa), \quad \frac{\partial \mu(\kappa, 0)}{\partial \tau} = 1, \quad 0 \leq \kappa \leq 1,$$

and BCs

$$\mu(0, \tau) = \sin(\tau), \quad \mu(1, \tau) = \sin(1) + \sin(\tau),$$

The analytic solution $\mu(\kappa, \tau) = \sin(\kappa) + \sin(\tau)$.

Fig. 3 compares exact and approximate solutions in two and three dimensions. Table 3 compares the absolute error for various time τ values, allowing one to assess the feasibility and correctness of the suggested strategy.

Table 3 Maximum absolute error (L_∞) are compared at different value of τ for Example 3

$(\varkappa)$	FWCM $\tau = 0.1$	FDM L_∞	FWCM $\tau = 0.3$
0.1	9.5006e-06	6.3650E-04	5.2432e-06
0.2	2.6463e-06	8.1077E-05	8.6101e-06
0.3	6.7253e-07	1.0162E-05	6.2513e-07
0.4	3.6211e-07	1.2712E-06	8.0321e-08
0.5	2.0602e-08	1.5894E-07	6.7551e-08
0.6	1.6214e-08	1.9868E-08	2.0362e-06

Example 4

$$\frac{\partial^2 \mu(\varkappa, \tau)}{\partial \tau^2} - \frac{\varkappa^2}{2} \frac{\partial^2 \mu(\varkappa, \tau)}{\partial \varkappa^2} = 0 \quad \tau \geq 0, \quad (38)$$

with ICs

$$\mu(\varkappa, 0) = \varkappa, \quad \frac{\partial \mu(\varkappa, 0)}{\partial \tau} = \varkappa^2, \quad 0 \leq \varkappa \leq 1,$$

BCs

$$\mu(0, \tau) = 0, \quad \mu(1, \tau) = 1 + 2 \sinh(\tau), \quad 0 \leq \tau \leq 1.$$

The exact solution is

$$\mu(\varkappa, \tau) = \varkappa + \varkappa^2 \sinh(\tau).$$

By the FWCM we obtain the system of equation given by

$$\sum_{l=0}^{2^{k-1}M} c_l [P_{l,2}(\varkappa_l) - \varkappa_l P_{i,2}(1)] = \frac{\varkappa_l}{2} u''(\varkappa_l, \tau_n) + \varkappa_l \sinh(\tau_{n+1}), \quad (39)$$

solved it using Newton's method to find the Fibonacci coefficient c_l .

In two and three dimensions, Fig. 4 compares the exact and approximative solutions. Table 4 provides the absolute error for various time $\tau = 0.01, 0.02, 0.03$ values, allowing one to assess the efficacy and precision of the suggested strategy. It is obvious that the results obtained with Fibonacci wavelets are very effective when compared to other techniques. The obtained results are presented in the form of graphs in two and three dimensions. The graphs clearly illustrate the convergence behavior, error distribution, and computational performance, offering visual confirmation of the method's effectiveness. Graphical analysis and interpretation of these results will enhance understanding and demonstrate the practical advantages of the proposed technique.

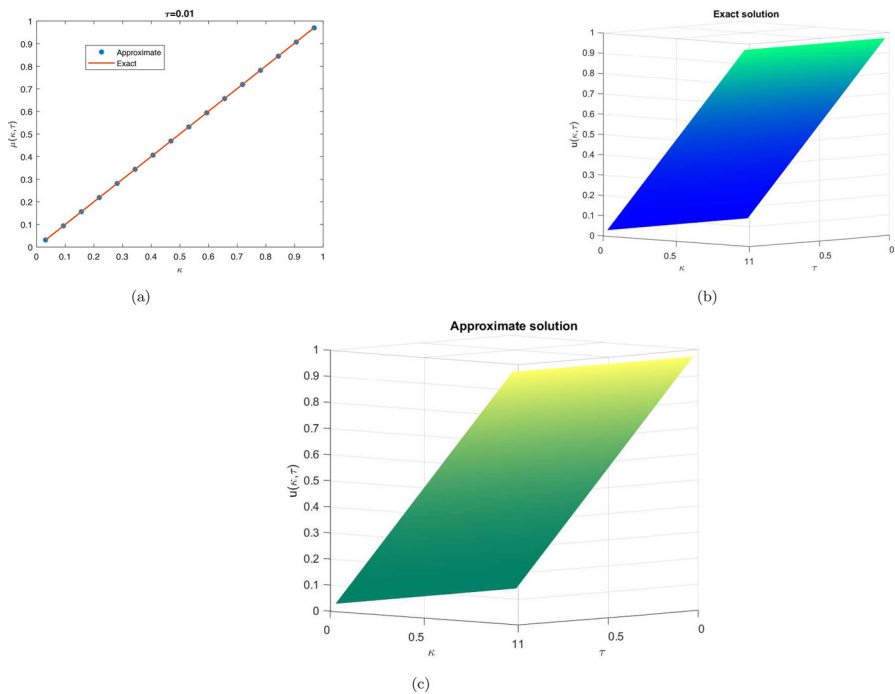


Fig. 4 **a** Comparison of approximate solution and exact solution **b** Approximate solution at different time levels

Table 4 For different value of time $\tau = 0.01, 0.02, 0.03$ the absolute error is calculated for Example 4

collocation point	$\tau=0.01$	$\tau=0.03$	$\tau=0.05$
0.1	3.4115e-08	8.7714e-07	1.3214e-06
0.2	5.1035e-07	8.7931e-06	2.2411e-06
0.3	4.0343e-06	1.2247e-06	2.5463e-05
0.4	1.4033e-06	1.1218e-05	3.2142e-05
0.5	2.4421e-05	1.3219e-05	3.3146e-04
0.6	1.3577e-05	1.3219e-05	3.2142e-04
0.7	1.1466e-05	9.5646e-04	2.3469e-04
0.8	7.1410e-05	5.7541e-04	1.2469e-04
0.9	3.1221e-05	8.5546e-04	6.2453e-04

6 Conclusion

This article introduced a Chen and Hesio-based technique for operational integration matrices of Fibonacci wavelets. By building the operational matrices of integration of the Fibonacci wavelet, the simplification was achieved, and as a result, the problems were reduced to sparse systems of linear equations, which are solved by Newton's

methods. Four test problems are used to illustrate how well the Fibonacci wavelet collocation approach works, and the outcomes are contrasted with those of other methods that have been published. Tables and figures presenting the numerical solutions allow us to see how well the Fibonacci solutions coincide with the actual solutions and FDM. When compared to other published methodologies, the results obtained are significantly better. The accuracy of the current solution grows as the values of k and M increase. This method offers physicists a strong and useful choice for effectively analyzing these kinds of PDEs and may be applied to similar physics problems.

Acknowledgements We would also like to thank Central University of Haryana for providing necessary facilities to carry out this research. Authors Kamal Shah and Thabet Abdeljawad are thankful to Prince Sultan University, Saudi Arabia for support through TAS research lab.

Author Contributions All the authors have contributed equally.

Funding Open access funding provided by Sefako Makgatho Health Sciences University. None.

Data availability None.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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