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A novel stability analysis of functional equation in neutrosophic normed spaces

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Abstract

The analysis of stability in functional equations (FEs) within neutrosophic normed spaces is a significant challenge due to the inherent uncertainties and complexities involved. This paper proposes a novel approach to address this challenge, offering a comprehensive framework for investigating stability properties in such contexts. Neutrosophic normed spaces are a generalization of traditional normed spaces that incorporate neutrosophic logic. By providing a systematic methodology for addressing stability concerns in neutrosophic normed spaces, our approach facilitates enhanced understanding and control of complex systems characterized by indeterminacy and uncertainty. The primary focus of this research is to propose a novel class of Euler-Lagrange additive FE and investigate its Ulam-Hyers stability in neutrosophic normed spaces. Direct and fixed point techniques are utilized to achieve the required results.

Mathematics Subject Classification: 39B52; 39B82

Keywords: Euler-Lagrange additive functional equations; Ulam-Hyers stability; Neutrosophic normed spaces; Fixed point theory

1 Introduction

Lotfi A. Zadeh, a mathematician and computer scientist, introduced the idea of fuzzy sets in his groundbreaking paper entitled “Fuzzy Sets,” published in 1965 [1]. Zadeh’s motivation was to address the limitations of classical set theory, which relies on crisp, well-defined boundaries for membership. The extension of fuzzy sets known as intuitionistic fuzzy sets (IFS) was initially proposed by Atanassov in 1983 [2, 3]. IFS provide a framework for dealing with uncertainty, vagueness, and hesitation more comprehensively than traditional fuzzy sets. A neutrosophic set is a mathematical concept introduced as an extension of classical set theory [4, 5]. Neutrosophic sets provide a way to handle indeterminacy, uncertainty, and incomplete information more flexibly.

Fuzzy normed spaces (FNS) are mathematical structures that extend the concept of normed spaces to include fuzzy numbers. Katsaras introduced the idea of FNS, a vector space equipped with a fuzzy norm, where the norm values are fuzzy numbers rather than real numbers [6]. The idea of intuitionistic fuzzy normed spaces (IFNS) was proposed in 2006 [7]. IFNS represent a fusion of concepts from fuzzy mathematics, intuitionistic fuzzy sets, and normed spaces, providing a versatile framework for handling uncertainty and

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imprecision in mathematical modeling and analysis. Neutrosophic normed linear spaces [8, 9] extend the ideas of neutrosophic sets to linear algebraic structures, offering a more expressive way to represent and handle uncertainty in vector spaces. Neutrosophic concepts have been applied across a wide range of mathematical fields such as groups, subgroups [10], vector spaces [11], homomorphisms of rings [12, 13], linear transformations [14], number theory [15], graph theory [16], measure theory, integral theory, probability theory [17], etc. Neutrosophic normed linear spaces find applications in various scientific and real-time applications like decision-making, control systems, optimization, image processing, pattern recognition, medical diagnosis, finance and risk management, information retrieval, and artificial intelligence [18–24].

Several mathematicians have derived the results of fixed point (F-P) theory using neutrosophic concepts. Ishtiaq et al. [25] proved F-P results in the framework of an orthogonal neutrosophic metric space. The common F-P results in the context of a neutrosophic metric space were established using contraction mapping [26]. Salama and Alblowi [27] studied the neutrosophic topological spaces, whereas Al-Omeri et al. [28] analyzed a neutrosophic cone metric space. Riaz et al. [29] discussed the F-P results for ξ -chainable neutrosophic and generalized neutrosophic cone metric spaces. Sharma et al. [30] studied the generalized summability using difference operators on neutrosophic normed spaces. One of the most prevalent stability theories was introduced by Ulam [31] and further developed by Hyers [32]. This theory is called the Ulam-Hyers stability and has applications in various branches of mathematics, including differential equations, functional analysis, and dynamic systems. Many researchers have worked on this theory and established the stability results in different normed spaces [33–40]. Recently, Agilan et al. introduced new kinds of FEs and established the Ulam-Hyers stability of the newly proposed equations in a variety of normed spaces [41–44]. Some related applications are also discussed in [45–48].

This article presents a novel category of Euler-Lagrange additive FE. The Ulam-Hyers stability of the newly introduced equation is analyzed in neutrosophic normed linear spaces using two techniques: direct and fixed point techniques. The stability analysis of the newly introduced equation holds significance due to the distinctive characteristics and potentially broad applications of neutrosophic normed spaces. It is noteworthy that, for the first time in the literature, the stability of an FE has been examined within neutrosophic normed spaces. The uniqueness of this endeavor underscores the importance of this research.

This study aims to achieve the following primary objectives:

- (a) Extend and enhance the current body of work on neutrosophic normed linear spaces.
- (b) Ascertain the uniqueness of the solution for the newly proposed Euler-Lagrange additive FE.
- (c) Derive the Ulam-Hyers stability of the newly proposed equation in neutrosophic normed linear spaces using the F-P method.

2 Definitions on neutrosophic normed spaces

Definition 2.1 The Seven-tuple $(\mathbb{A}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a, *, \diamond, \oslash)$ is said to be a neutrosophic normed space (for short, NNS) if $\mathbb{A}$ is a vector space, $*$ is a continuous κ -norm, $\diamond$ and $\oslash$ is a continuous κ -conorm, and $\mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a$ are fuzzy sets on $\mathbb{A} \times (0, \infty)$ satisfying the following conditions. For every $p, q \in \mathbb{A}$ and $s, \kappa > 0$,

- (A1) $\mathfrak{A}_a(p, \kappa) + \mathfrak{B}_a(p, \kappa) + \mathfrak{C}_a(p, \kappa) \leq 3$,
- (A2) $0 \leq \mathfrak{A}_a(p, \kappa) \leq 1, 0 \leq \mathfrak{B}_a(p, \kappa) \leq 1, 0 \leq \mathfrak{C}_a(p, \kappa) \leq 1$,
- (A3) $\mathfrak{A}_a(p, \kappa) > 0$,
- (A4) $\mathfrak{A}_a(p, \kappa) = 1$, if and only if $p = 0$.
- (A5) $\mathfrak{A}_a(\alpha p, \kappa) = \mathfrak{A}_a(p, \frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
- (A6) $\mathfrak{A}_a(p, \kappa) * \mathfrak{A}_a(q, s) \leq \mathfrak{A}_a(p + q, \kappa + s)$,
- (A7) $\mathfrak{A}_a(p, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (A8) $\lim_{\kappa \rightarrow \infty} \mathfrak{A}_a(p, \kappa) = 1$ and $\lim_{\kappa \rightarrow 0} \mathfrak{A}_a(p, \kappa) = 0$,
- (A9) $\mathfrak{B}_a(p, \kappa) < 1$,
- (A10) $\mathfrak{B}_a(p, \kappa) = 0$, if and only if $p = 0$.
- (A11) $\mathfrak{B}_a(\alpha p, \kappa) = \mathfrak{B}_a(p, \frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
- (A12) $\mathfrak{B}_a(p, \kappa) \diamond \mathfrak{B}_a(q, s) \geq \mathfrak{B}_a(p + q, \kappa + s)$,
- (A13) $\mathfrak{B}_a(p, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (A14) $\lim_{\kappa \rightarrow \infty} \mathfrak{B}_a(p, \kappa) = 0$ and $\lim_{\kappa \rightarrow 0} \mathfrak{B}_a(p, \kappa) = 1$,
- (A15) $\mathfrak{C}_a(p, \kappa) < 1$,
- (A16) $\mathfrak{C}_a(p, \kappa) = 0$, if and only if $p = 0$.
- (A17) $\mathfrak{C}_a(\alpha p, \kappa) = \mathfrak{C}_a(p, \frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
- (A18) $\mathfrak{C}_a(p, \kappa) \oslash \mathfrak{C}_a(q, s) \geq \mathfrak{C}_a(p + q, \kappa + s)$,
- (A19) $\mathfrak{C}_a(p, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (A20) $\lim_{\kappa \rightarrow \infty} \mathfrak{C}_a(p, \kappa) = 0$ and $\lim_{\kappa \rightarrow 0} \mathfrak{C}_a(p, \kappa) = 1$.

3 Stability results: direct method

The newly proposed Euler-Lagrange additive FE is as follows:

$$\begin{aligned} & \mathfrak{T}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) + \mathcal{P}_1 \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) \\ & + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{Q}_1(\mathcal{W}_a - \mathcal{X}_a)) \\ & = \mathfrak{T}_1(\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \end{aligned} \quad (3.1)$$

where $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1 \in \mathbb{R}$ with $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1 \neq 0$ and $\mathfrak{T}_1 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1 \neq 0$ in neutrosophic normed space using direct and F-P methods.

Assume that $(\mathcal{M}, \mathfrak{A}'_a, \mathfrak{B}'_a, \mathfrak{C}'_a)$ is a neutrosophic normed linear space and $(\mathcal{M}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$ is a neutrosophic Banach space. Let $\mathcal{L}$ be a linear space. Then,

$$\begin{aligned} \mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a) &= \mathfrak{T}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) + \mathcal{P}_1 \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) \\ & + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{Q}_1(\mathcal{W}_a - \mathcal{X}_a)) \\ & - \mathfrak{T}_1(\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \end{aligned}$$

where $\mathfrak{T}_1 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1, \mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1 \in \mathbb{R}$ and $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1 \neq 0 \forall \mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a \in \mathcal{L}$.

Theorem 3.1 Let $N : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{M}$ be a mapping with the condition $0 < (\frac{\mathfrak{X}_a}{\mathfrak{T}_1})^\eta < 1$, then

$$\left. \begin{aligned} & \mathfrak{A}'_a(N(\mathfrak{T}_1^{m\eta} \mathcal{X}_a, \mathfrak{T}_1^{m\eta} \mathcal{V}_a, \mathfrak{T}_1^{m\eta} \mathcal{W}_a), v) \geq \mathfrak{A}'_a(\mathcal{X}_a^{m\eta} N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ & \mathfrak{B}'_a(N(\mathfrak{T}_1^{m\eta} \mathcal{X}_a, \mathfrak{T}_1^{m\eta} \mathcal{V}_a, \mathfrak{T}_1^{m\eta} \mathcal{W}_a), v) \leq \mathfrak{B}'_a(\mathcal{X}_a^{m\eta} N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ & \mathfrak{C}'_a(N(\mathfrak{T}_1^{m\eta} \mathcal{X}_a, \mathfrak{T}_1^{m\eta} \mathcal{V}_a, \mathfrak{T}_1^{m\eta} \mathcal{W}_a), v) \leq \mathfrak{C}'_a(\mathcal{X}_a^{m\eta} N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) \end{aligned} \right\} \quad (3.2)$$

and

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{A}'_a(N(\mathfrak{T}_1^{\eta n} \mathcal{X}_a, \mathfrak{T}_1^{\eta n} \mathcal{V}_a, \mathfrak{T}_1^{\eta n} \mathcal{W}_a), a^{\eta n} v) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{B}'_a(N(\mathfrak{T}_1^{\eta n} \mathcal{X}_a, \mathfrak{T}_1^{\eta n} \mathcal{V}_a, \mathfrak{T}_1^{\eta n} \mathcal{W}_a), a^{\eta n} v) &= 0, \\ \lim_{n \rightarrow \infty} \mathfrak{C}'_a(N(\mathfrak{T}_1^{\eta n} \mathcal{X}_a, \mathfrak{T}_1^{\eta n} \mathcal{V}_a, \mathfrak{T}_1^{\eta n} \mathcal{W}_a), a^{\eta n} v) &= 0. \end{aligned} \right\} \quad (3.3)$$

Assume that a mapping $\mathcal{A}_1 : \mathcal{L} \rightarrow \mathcal{M}$ satisfies the inequality

$$\left. \begin{aligned} \mathfrak{A}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\geq \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ \mathfrak{B}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\leq \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ \mathfrak{C}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\leq \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) \end{aligned} \right\} \quad (3.4)$$

and $\exists$ unique additive function $\mathcal{A}_1 : \mathcal{L} \rightarrow \mathcal{M}$

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 | \mathfrak{T}_1 - \mathcal{X}_a | v), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 | \mathfrak{T}_1 - \mathcal{X}_a | v), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 | \mathfrak{T}_1 - \mathcal{X}_a | v) \end{aligned} \right\} \quad (3.5)$$

with the conditions $\eta \in \{1, -1\}$, where $\mathfrak{T}_1 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1$.

Proof Let us consider $(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a)$ by $(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a)$ in (3.4), we reach

$$\left. \begin{aligned} \mathfrak{A}_a(\mathfrak{T}_1 \mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a) - \mathfrak{T}_1^2 \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{B}_a(\mathfrak{T}_1 \mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a) - \mathfrak{T}_1^2 \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{C}_a(\mathfrak{T}_1 \mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a) - \mathfrak{T}_1^2 \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v). \end{aligned} \right\} \quad (3.6)$$

By applying the conditions of neutrosophic normed space, we arrive

$$\left. \begin{aligned} \mathfrak{A}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\geq \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{B}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\leq \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{C}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\leq \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v). \end{aligned} \right\} \quad (3.7)$$

Replacing $\mathcal{X}_a$ by $\mathfrak{T}_1^n \mathcal{X}_a$ in (3.7), we get

$$\left. \begin{aligned} \mathfrak{A}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\geq \mathfrak{A}'_a(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), v), \\ \mathfrak{B}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\leq \mathfrak{B}'_a(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), v), \\ \mathfrak{C}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v}{\mathfrak{T}_1^2}\right) &\leq \mathfrak{C}'_a(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), v). \end{aligned} \right\} \quad (3.8)$$

Using (3.8) and conditions of neutrosophic normed space, we arrive

$$\left. \begin{aligned} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v}{\mathfrak{T}_1^{n+2}} \right) &\geq \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^n} \right), \\ \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v}{\mathfrak{T}_1^{n+2}} \right) &\leq \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^n} \right), \\ \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v}{\mathfrak{T}_1^{n+2}} \right) &\leq \mathfrak{C}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^n} \right). \end{aligned} \right\} \quad (3.9)$$

Let v by $\mathcal{X}_a^n v$ in (3.9), then

$$\left. \begin{aligned} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v \cdot \mathcal{X}_a^n}{\mathfrak{T}_1^{n+2}} \right) &\geq \mathfrak{A}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v \cdot \mathcal{X}_a^n}{\mathfrak{T}_1^{n+2}} \right) &\leq \mathfrak{B}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}, \frac{v \cdot \mathcal{X}_a^n}{\mathfrak{T}_1^{n+2}} \right) &\leq \mathfrak{C}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v). \end{aligned} \right\} \quad (3.10)$$

It is easy to see that

$$\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a) = \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}. \quad (3.11)$$

From (3.10) and (3.11), we reach

$$\left. \begin{aligned} &\mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ &= \mathfrak{A}_a \left(\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right), \\ &\mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ &= \mathfrak{B}_a \left(\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right), \\ &\mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ &= \mathfrak{C}_a \left(\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right). \end{aligned} \right\} \quad (3.12)$$

From (3.12), we arrive

$$\left. \begin{aligned} & \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \geq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right), \\ & \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \leq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right), \\ & \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \leq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}+1} \mathcal{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^{\mathcal{J}} \mathcal{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}, \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right), \end{aligned} \right\} \quad (3.13)$$

where

$$\begin{aligned} \prod_{\mathcal{J}=0}^{n-1} Q_j &= Q_1 * Q_2 * \cdots * Q_n \quad \text{and} \quad \prod_{\mathcal{J}=0}^{n-1} R_j = R_1 \diamond R_2 \diamond \cdots \diamond R_n \quad \text{and} \\ \prod_{\mathcal{J}=0}^{n-1} S_j &= S_1 \oslash S_2 \oslash \cdots \oslash S_n. \end{aligned}$$

Using the above conditions, we have

$$\left. \begin{aligned} & \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \geq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v) = \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ & \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \leq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v) = \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ & \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{\mathcal{J}+2}} \right) \\ & \leq \prod_{\mathcal{J}=0}^{n-1} \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v) = \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v). \end{aligned} \right\} \quad (3.14)$$

Considering $\mathcal{X}_a$ by $\mathfrak{T}_1^m \mathcal{X}_a$ in (3.14) and dividing by $\mathfrak{T}_1^m$, we get

$$\left. \begin{aligned} & \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \\ & \geq \mathfrak{A}'_a(N(\mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a), v) \\ & = \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^m} \right), \\ & \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \\ & \leq \mathfrak{B}'_a(N(\mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a), v) \\ & = \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^m} \right), \\ & \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{C}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \\ & \leq \mathfrak{C}'_a(N(\mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a, \mathfrak{T}_1^m \mathcal{X}_a), v) \\ & = \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\mathcal{X}_a^m} \right). \end{aligned} \right\}$$

Replacing v by $\mathcal{X}_a^m v$ in the above inequality, we have

$$\left. \begin{aligned} & \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}+m} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \geq \mathfrak{A}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ & \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}+m} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \leq \mathfrak{B}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \\ & \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, \sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}+m} v}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}} \right) \leq \mathfrak{C}'_a(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), v), \end{aligned} \right\} \quad (3.15)$$

$$\left. \begin{aligned} & \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, v \right) \geq \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=m}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right), \\ & \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, v \right) \leq \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=m}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right), \\ & \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^{n+m} \mathcal{X}_a)}{\mathfrak{T}_1^{(n+m)}} - \frac{\mathcal{A}_1(\mathfrak{T}_1^m \mathcal{X}_a)}{\mathfrak{T}_1^m}, v \right) \leq \mathfrak{C}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=m}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right). \end{aligned} \right\} \quad (3.16)$$

Since $0 < \mathcal{X}_a < 1$ and $\sum_{\mathcal{J}=0}^n (\frac{\mathcal{X}_a}{\mathfrak{T}_1})^{\mathcal{J}} < \infty$. The sequence $\{\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n}\}$ is Cauchy in $(Y, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$. Since $(Y, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$ is a complete NSN-space, this sequence converges to some point $\mathcal{A}_1(\mathcal{X}_a) \in Y$. Defining $\mathcal{A}_1 : \mathcal{L} \rightarrow \mathcal{M}$ by

$$\lim_{n \rightarrow \infty} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) = 1,$$

$$\lim_{n \rightarrow \infty} \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) = 0,$$

$$\lim_{n \rightarrow \infty} \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) = 0.$$

Finally

$$\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} \xrightarrow{\text{NSN}} \mathcal{A}_1(\mathcal{X}_a), \quad \text{as } n \rightarrow \infty.$$

Consider $m = 0$ in (3.16), then

$$\left. \begin{aligned} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &\geq \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right), \\ \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &\leq \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right), \\ \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &\leq \mathfrak{C}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathcal{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}} \right). \end{aligned} \right\} \quad (3.17)$$

As $n \rightarrow \infty$ in (3.17) and

$$\left. \begin{aligned} \mathfrak{A}_a (\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), t) &\geq \mathfrak{A}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v (\mathfrak{T}_1 - \mathcal{X}_a)), \\ \mathfrak{B}_a (\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), t) &\leq \mathfrak{B}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v (\mathfrak{T}_1 - \mathcal{X}_a)), \\ \mathfrak{C}_a (\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), t) &\leq \mathfrak{C}'_a (N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v (\mathfrak{T}_1 - \mathcal{X}_a)). \end{aligned} \right\} \quad (3.18)$$

Finally, $\mathcal{A}_1$ satisfies (3.1), taking $(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a)$ by $(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a)$ in (3.4).

$$\left. \begin{aligned} \mathfrak{A}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{J}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), v \right) &\geq \mathfrak{A}'_a (N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \mathfrak{T}_1^n v), \\ \mathfrak{B}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{J}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), v \right) &\leq \mathfrak{B}'_a (N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \mathfrak{T}_1^n v), \\ \mathfrak{C}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{J}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), v \right) &\leq \mathfrak{C}'_a (N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \mathfrak{T}_1^n v). \end{aligned} \right\} \quad (3.19)$$

Here,

$$\begin{aligned} &\mathfrak{A}_a (\mathcal{P}_1 \mathcal{A}_1 (\mathcal{Q}_1 \mathcal{R}_1 (\mathcal{X}_a - \mathcal{V}_a)) + \mathcal{Q}_1 \mathcal{A}_1 (\mathcal{P}_1 \mathcal{R}_1 (\mathcal{V}_a - \mathcal{W}_a)) \\ &\quad + \mathcal{R}_1 \mathcal{A}_1 (\mathcal{P}_1 \mathcal{Q}_1 (\mathcal{W}_a - \mathcal{X}_a)) + \mathfrak{T}_1 \mathcal{A}_1 (\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) \\ &\quad - \mathfrak{T}_1 (\mathcal{P}_1 \mathcal{A}_1 (\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1 (\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1 (\mathcal{W}_a))) \\ &\geq \mathfrak{A}_a \left(\mathcal{P}_1 \mathcal{A}_1 (\mathcal{Q}_1 \mathcal{R}_1 (\mathcal{X}_a - \mathcal{V}_a)) - \frac{\mathcal{P}_1}{\mathfrak{T}_1^n} \mathcal{A}_1 (\mathcal{Q}_1 \mathcal{R}_1 (\mathcal{X}_a - \mathcal{V}_a)), \frac{v}{6} \right) \\ &\quad * \mathfrak{A}_a \left(\mathcal{Q}_1 \mathcal{A}_1 (\mathcal{P}_1 \mathcal{R}_1 (\mathcal{V}_a - \mathcal{W}_a)) - \frac{\mathcal{Q}_1}{\mathfrak{T}_1^n} \mathcal{A}_1 (\mathcal{P}_1 \mathcal{R}_1 (\mathcal{V}_a - \mathcal{W}_a)), \frac{v}{6} \right) \end{aligned}$$

$$\begin{aligned}
& * \mathfrak{A}_a \left(\mathfrak{T}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a), \frac{\nu}{6} \right) \\
& * \mathfrak{A}_a \left(-\mathfrak{T}_1 (\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)) \right. \\
& \left. + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{\nu}{6} \right) \\
& * \mathfrak{A}_a \left(\frac{\mathcal{P}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) + \frac{\mathcal{Q}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) \right. \\
& \left. + \frac{\mathcal{R}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{Q}_1(\mathcal{W}_a - \mathcal{X}_a)) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) \right. \\
& \left. - \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{\nu}{6} \right) \tag{3.20}
\end{aligned}$$

and

$$\begin{aligned}
& \mathfrak{B}_a(P_1 A_1(Q_1 R_1(\mathcal{X}_a - \mathcal{V}_a)) + Q_1 A_1(P_1 R_1(\mathcal{V}_a - \mathcal{W}_a))) \\
& + R_1 A_1(P_1 Q_1(\mathcal{W}_a - \mathcal{X}_a)) + \mathfrak{T}_1 A_1(P_1 \mathcal{X}_a + Q_1 \mathcal{V}_a + R_1 \mathcal{W}_a) \\
& - \mathfrak{T}_1(P_1 A_1(\mathcal{X}_a) + Q_1 A_1(\mathcal{V}_a) + R_1 A_1(\mathcal{W}_a))) \\
& \geq \mathfrak{B}_a\left(P_1 A_1(Q_1 R_1(\mathcal{X}_a - \mathcal{V}_a)) - \frac{P_1}{\mathfrak{T}_1^n} \mathcal{A}_1(Q_1 R_1(\mathcal{X}_a - \mathcal{V}_a)), \frac{v}{6}\right) \\
& \diamond \mathfrak{B}_a\left(Q_1 A_1(P_1 R_1(\mathcal{V}_a - \mathcal{W}_a)) - \frac{Q_1}{\mathfrak{T}_1^n} \mathcal{A}_1(P_1 R_1(\mathcal{V}_a - \mathcal{W}_a)), \frac{v}{6}\right) \\
& \diamond \mathfrak{B}_a\left(\mathfrak{T}_1 A_1(P_1 \mathcal{X}_a + Q_1 \mathcal{V}_a + R_1 \mathcal{W}_a) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(P_1 \mathcal{X}_a + Q_1 \mathcal{V}_a + R_1 \mathcal{W}_a), \frac{v}{6}\right) \\
& \diamond \mathfrak{B}_a\left(-\mathfrak{T}_1(P_1 A_1(\mathcal{X}_a) + Q_1 A_1(\mathcal{V}_a) + R_1 A_1(\mathcal{W}_a)) \right. \\
& \left. + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (P_1 \mathcal{A}_1(\mathcal{X}_a) + Q_1 \mathcal{A}_1(\mathcal{V}_a) + R_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{v}{6}\right) \\
& \diamond \mathfrak{B}_a\left(\frac{P_1}{\mathfrak{T}_1^n} \mathcal{A}_1(Q_1 R_1(\mathcal{X}_a - \mathcal{V}_a)) + \frac{Q_1}{\mathfrak{T}_1^n} \mathcal{A}_1(P_1 R_1(\mathcal{V}_a - \mathcal{W}_a)) \right. \\
& \left. + \frac{R_1}{\mathfrak{T}_1^n} \mathcal{A}_1(P_1 Q_1(\mathcal{W}_a - \mathcal{X}_a)) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(P_1 \mathcal{X}_a + Q_1 \mathcal{V}_a + R_1 \mathcal{W}_a) \right. \\
& \left. - \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (P_1 \mathcal{A}_1(\mathcal{X}_a) + Q_1 \mathcal{A}_1(\mathcal{V}_a) + R_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{v}{6}\right) \tag{3.21}
\end{aligned}$$

and

$$\begin{aligned} & \mathfrak{C}_a(\mathcal{P}_1 \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) \\ & \quad + \mathcal{R}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{Q}_1(\mathcal{W}_a - \mathcal{X}_a)) + \mathfrak{T}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) \\ & \quad - \mathfrak{T}_1(\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a))) \\ & \geq \mathfrak{C}_a\left(\mathcal{P}_1 \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) - \frac{\mathcal{P}_1}{\mathfrak{T}_1^{\frac{1}{\alpha}}} \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)), \frac{v}{6}\right) \end{aligned}$$

$$\begin{aligned}
& \odot \mathfrak{C}_a \left(\mathcal{Q}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) - \frac{\mathcal{Q}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)), \frac{v}{6} \right) \\
& \odot \mathfrak{C}_a \left(\mathfrak{T}_1 \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a), \frac{v}{6} \right) \\
& \odot \mathfrak{C}_a \left(-\mathfrak{T}_1(\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)) \right. \\
& \quad \left. + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{v}{6} \right) \\
& \odot \mathfrak{C}_a \left(\frac{\mathcal{P}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{Q}_1 \mathcal{R}_1(\mathcal{X}_a - \mathcal{V}_a)) + \frac{\mathcal{Q}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{R}_1(\mathcal{V}_a - \mathcal{W}_a)) \right. \\
& \quad \left. + \frac{\mathcal{R}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{Q}_1(\mathcal{W}_a - \mathcal{X}_a)) + \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} \mathcal{A}_1(\mathcal{P}_1 \mathcal{X}_a + \mathcal{Q}_1 \mathcal{V}_a + \mathcal{R}_1 \mathcal{W}_a) \right. \\
& \quad \left. - \frac{\mathfrak{T}_1}{\mathfrak{T}_1^n} (\mathcal{P}_1 \mathcal{A}_1(\mathcal{X}_a) + \mathcal{Q}_1 \mathcal{A}_1(\mathcal{V}_a) + \mathcal{R}_1 \mathcal{A}_1(\mathcal{W}_a)), \frac{v}{6} \right). \tag{3.22}
\end{aligned}$$

Also,

$$\left. \begin{aligned}
& \lim_{n \rightarrow \infty} \mathfrak{A}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{Z}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \frac{v}{6} \right) = 1, \\
& \lim_{n \rightarrow \infty} \mathfrak{B}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{Z}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \frac{v}{6} \right) = 0, \\
& \lim_{n \rightarrow \infty} \mathfrak{C}_a \left(\frac{1}{\mathfrak{T}_1^n} \mathfrak{Z}(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{V}_a, \mathfrak{T}_1^n \mathcal{W}_a), \frac{v}{6} \right) = 0.
\end{aligned} \right\} \tag{3.23}$$

To prove uniqueness

$$\begin{aligned}
& \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) \\
& \geq \mathfrak{A}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right) * \mathfrak{A}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}'_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right) \\
& \geq \mathfrak{A}'_a \left(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1}}{2} |\mathfrak{T}_1 - \mathcal{X}_a| \right) \\
& \geq \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right), \\
& \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) \\
& \leq \mathfrak{B}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right) \diamond \mathfrak{B}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}'_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right) \\
& \leq \mathfrak{B}'_a \left(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1}}{2} |\mathfrak{T}_1 - \mathcal{X}_a| \right) \\
& \leq \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right), \\
& \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) \\
& \leq \mathfrak{C}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right) \diamond \mathfrak{C}_a \left(\mathcal{A}_1(\mathfrak{T}_1^n \mathcal{X}_a) - \mathcal{A}'_1(\mathfrak{T}_1^n \mathcal{X}_a), \frac{v \cdot \mathfrak{T}_1^n}{2} \right)
\end{aligned}$$

$$\begin{aligned} &\leq \mathfrak{C}'_a \left(N(\mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a, \mathfrak{T}_1^n \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2} \right) \\ &\leq \mathfrak{C}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \mathcal{X}_a^n} = \infty$,

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{A}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{B}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right) &= 0, \\ \lim_{n \rightarrow \infty} \mathfrak{C}'_a \left(N(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \frac{v \mathfrak{T}_1^{n+1} |\mathfrak{T}_1 - \mathcal{X}_a|}{2 \cdot \mathcal{X}_a^n} \right) &= 0. \end{aligned} \right\}$$

Hence,

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) &= 1, \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) &= 0, \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}'_1(\mathcal{X}_a), v) &= 0. \end{aligned} \right\}$$

Thus, $\mathcal{A}_1(\mathcal{X}_a) = \mathcal{A}'_1(\mathcal{X}_a)$. Hence, $\mathcal{A}_1(\mathcal{X}_a)$ is unique.

Method 2: Assume that $\eta = -1$. Substituting p by $\frac{\mathcal{X}_a}{\mathfrak{T}_1}$ in (3.6) gives

$$\left. \begin{aligned} \mathfrak{A}_a \left(\mathfrak{T}_1 \mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1^2 \omega \left(\frac{\mathcal{X}_a}{\mathfrak{T}_1} \right), v \right) &\geq \mathfrak{A}'_a \left(N \left(\frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2} \right), v \right), \\ \mathfrak{B}_a \left(\mathfrak{T}_1 \mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1^2 \omega \left(\frac{\mathcal{X}_a}{\mathfrak{T}_1} \right), v \right) &\leq \mathfrak{B}'_a \left(N \left(\frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2} \right), v \right), \\ \mathfrak{C}_a \left(\mathfrak{T}_1 \mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1^2 \omega \left(\frac{\mathcal{X}_a}{\mathfrak{T}_1} \right), v \right) &\leq \mathfrak{C}'_a \left(N \left(\frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2}, \frac{\mathcal{X}_a}{2} \right), v \right). \end{aligned} \right\} \quad (3.24)$$

□

Corollary 3.2 *Let $\mathcal{A}_1$ be an approximately additive mapping from $(Z, \mathfrak{A}'_a, \mathfrak{B}'_a, \mathfrak{C}'_a)$ in a neutrosophic normed space and $(\mathcal{M}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$ be a neutrosophic Banach space that satisfies the inequality*

$$\begin{aligned} &\mathfrak{A}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{Y}_a, \mathcal{W}_a), v) \\ &\geq \begin{cases} \mathfrak{A}'_a(\mathcal{S}, v), \\ \mathfrak{A}'_a(\mathcal{S}(\|\mathcal{X}_a\|^{\mathfrak{E}} + \|\mathcal{Y}_a\|^{\mathfrak{F}} + \|\mathcal{W}_a\|^{\mathfrak{G}}), v), & \mathfrak{E}, \mathfrak{F}, \mathfrak{G} \neq 1, \\ \mathfrak{A}'_a(\mathcal{S}(\|\mathcal{X}_a\|^{\mathfrak{E}} \|\mathcal{Y}_a\|^{\mathfrak{F}} \|\mathcal{W}_a\|^{\mathfrak{G}}), v), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1, \\ \mathfrak{A}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^{\mathfrak{E}} \|\mathcal{Y}_a\|^{\mathfrak{F}} \|\mathcal{W}_a\|^{\mathfrak{G}} \\ + (\|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{Y}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}})\}, v), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1, \end{cases} \end{aligned} \quad (3.25)$$

$$\mathfrak{B}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{Y}_a, \mathcal{W}_a), \nu) \quad (3.26)$$

$$\leq \begin{cases} \mathfrak{B}'_a(\mathcal{S}, \nu), \\ \mathfrak{B}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{Y}_a\|^\mathfrak{F} + \|\mathcal{W}_a\|^\mathfrak{G}), \nu), & \mathfrak{E}, \mathfrak{F}, \mathfrak{G} \neq 1, \\ \mathfrak{B}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{X}_a\|^\mathfrak{F} \|\mathcal{Y}_a\|^\mathfrak{G}), \nu), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1, \\ \mathfrak{B}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{Y}_a\|^\mathfrak{F} \|\mathcal{W}_a\|^\mathfrak{G} \\ + (\|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{Y}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}})\}, \nu), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1, \end{cases} \quad (3.27)$$

$$\mathfrak{C}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{Y}_a, \mathcal{W}_a), \nu) \quad (3.28)$$

$$\leq \begin{cases} \mathfrak{C}'_a(\mathcal{S}, \nu), \\ \mathfrak{C}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{Y}_a\|^\mathfrak{F} + \|\mathcal{W}_a\|^\mathfrak{G}), \nu), & \mathfrak{E}, \mathfrak{F}, \mathfrak{G} \neq 1, \\ \mathfrak{C}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{X}_a\|^\mathfrak{F} \|\mathcal{Y}_a\|^\mathfrak{G}), \nu), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1, \\ \mathfrak{C}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{Y}_a\|^\mathfrak{F} \|\mathcal{W}_a\|^\mathfrak{G} \\ + (\|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{Y}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}})\}, \nu), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1 \end{cases} \quad (3.29)$$

such that

$$\mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), \nu)$$

$$\geq \begin{cases} \mathfrak{A}'_a(\mathcal{S}, \mathfrak{T}_1 \nu |\mathfrak{T}_1 - 1|), \\ \mathfrak{A}'_a([\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \mathfrak{T}_1 \nu [|\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{E}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{F}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{G}|]), \\ \mathfrak{A}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \mathfrak{T}_1 \nu |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|), \\ \mathfrak{A}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ + [\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \mathfrak{T}_1 \nu |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|), \end{cases}$$

$$\mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), \nu)$$

$$\leq \begin{cases} \mathfrak{B}'_a(\mathcal{S}, \mathfrak{T}_1 \nu |\mathfrak{T}_1 - 1|), \\ \mathfrak{B}'_a([\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \mathfrak{T}_1 \nu [|\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{E}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{F}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{G}|]), \\ \mathfrak{B}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \mathfrak{T}_1 \nu |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|), \\ \mathfrak{B}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ + [\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \mathfrak{T}_1 \nu |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|), \end{cases} \quad (3.30)$$

$$\begin{aligned} & \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \\ & \leq \begin{cases} \mathfrak{C}'_a(\mathcal{S}, \mathfrak{T}_1 v |\mathfrak{T}_1 - 1|), \\ \mathfrak{C}'_a([\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \quad \mathfrak{T}_1 v [|\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{E}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{F}| + |\mathfrak{T}_1 - \mathfrak{T}_1^\mathfrak{G}|]), \\ \mathfrak{C}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \mathfrak{T}_1 v |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|), \\ \mathfrak{C}'_a(\mathcal{S} \|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} |\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ \quad + [\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} |\mathfrak{T}_1|^\mathfrak{E} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{F} |\mathfrak{T}_1|^\mathfrak{F} + \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{G} |\mathfrak{T}_1|^\mathfrak{G}], \\ \quad \mathfrak{T}_1 v |\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|). \end{cases} \end{aligned}$$

Proof Let

$$N(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a) = \begin{cases} \mathcal{S}, \\ \mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{V}_a\|^\mathfrak{F} + \|\mathcal{W}_a\|^\mathfrak{G}), \\ \mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{V}_a\|^\mathfrak{F} \|\mathcal{W}_a\|^\mathfrak{G}, \\ \mathcal{S} \{ \|\mathcal{X}_a\|^\mathfrak{E} \|\mathcal{V}_a\|^\mathfrak{F} \|\mathcal{W}_a\|^\mathfrak{G} \\ \quad + (\|\mathcal{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{V}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathcal{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}) \}, \end{cases}$$

and

$$\mathcal{X}_a = \begin{cases} \mathfrak{T}_1^0, \\ \mathfrak{T}_1^A + \mathfrak{T}_1^B + \mathfrak{T}_1^C, \\ \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \\ \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}. \end{cases}$$

□

4 Stability results: fixed point method [49]

Theorem 4.1 Let $N : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{M}$ such that

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{A}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}^n v) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{B}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}^n v) &= 0, \\ \lim_{n \rightarrow \infty} \mathfrak{C}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}^n v) &= 0, \end{aligned} \right\} \quad (4.1)$$

where

$$\mathfrak{D}_i = \begin{cases} \mathfrak{T}_1 & \text{if } i = 0, \\ \frac{1}{\mathfrak{T}_1} & \text{if } i = 1, \end{cases} \quad (4.2)$$

then

$$\left. \begin{aligned} \mathfrak{A}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\geq \mathfrak{A}'_a(K(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ \mathfrak{B}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\leq \mathfrak{B}'_a(K(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v), \\ \mathfrak{C}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) &\leq \mathfrak{C}'_a(K(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v). \end{aligned} \right\} \quad (4.3)$$

If $L = L(i)$ and

$$\mathcal{A}_1(\mathcal{X}_a) = \frac{1}{\mathfrak{T}_1} K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), \quad (4.4)$$

and

$$\left. \begin{aligned} \mathfrak{A}'_a\left(L \frac{\mathcal{A}_1(\mathfrak{D}_i \mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{B}'_a\left(L \frac{\mathcal{A}_1(\mathfrak{D}_i \mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{C}'_a\left(L \frac{\mathcal{A}_1(\mathfrak{D}_i \mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), v) \end{aligned} \right\} \quad (4.5)$$

also

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L} v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L} v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L} v\right). \end{aligned} \right\} \quad (4.6)$$

Proof Let

$$\begin{aligned} \mathcal{S} &= \{h_1 \mid h_1 : \mathcal{X}_a \longrightarrow Y, \mathcal{A}_1(0) = 0\}, \\ d(h_1, f_1) &= \inf \left\{ L \in (0, \infty) : \left\{ \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), v > 0, \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), v > 0, \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), v > 0 \end{aligned} \right\} \right\}. \end{aligned} \quad (4.7)$$

Hence, by (4.7)

$$\inf \left\{ L \in (0, \infty) : \begin{cases} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{A}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), \mathfrak{D}_i v), \\ \mathfrak{A}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{A}_a(J\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{B}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), \mathfrak{D}_i v), \\ \mathfrak{B}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{B}_a(J\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{C}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), \mathfrak{D}_i v), \\ \mathfrak{C}_a\left(\frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a) - \frac{1}{\mathfrak{D}_i}\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a), v\right) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{C}_a(J\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv) \end{cases} \right\}$$

after that

$$\inf \left\{ 1 \in (0, \infty) : \begin{cases} \mathfrak{A}_a(\mathcal{A}_1(\mathfrak{T}_1\mathcal{X}_a) - \mathfrak{T}_1\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathfrak{T}_1\mathcal{X}_a) - \mathfrak{T}_1\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathfrak{T}_1\mathcal{X}_a) - \mathfrak{T}_1\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v) \end{cases} \right\}. \quad (4.8)$$

Assuming $i = 0$,

$$\inf \left\{ L^{1-0} \in (0, \infty) : \begin{cases} \mathfrak{A}_a(\mathcal{A}_1(\mathfrak{T}_1\mathcal{X}_a) - \mathfrak{T}_1\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v), \\ \mathfrak{A}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1\mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), v\right) \geq \mathfrak{A}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1^2 v), \\ \mathfrak{A}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{A}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{A}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv) \end{cases} \right\}, \quad (4.9)$$

$$\inf_{L^{1-0} \in (0, \infty)} : \left\{ \begin{array}{l} \mathfrak{B}_a(\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a) - \mathfrak{T}_1 \mathcal{A}_1(\mathcal{X}_a), t) \leq \mathfrak{B}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v), \\ \mathfrak{B}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), v\right) \leq \mathfrak{B}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1^2 v), \\ \mathfrak{B}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{B}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{B}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv) \end{array} \right\}, \quad (4.10)$$

$$\inf_{L^{1-0} \in (0, \infty)} : \left\{ \begin{array}{l} \mathfrak{C}_a(\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a) - \mathfrak{T}_1 \mathcal{A}_1(\mathcal{X}_a), t) \leq \mathfrak{C}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1 v), \\ \mathfrak{C}_a\left(\frac{\mathcal{A}_1(\mathfrak{T}_1 \mathcal{X}_a)}{\mathfrak{T}_1} - \mathcal{A}_1(\mathcal{X}_a), v\right) \leq \mathfrak{C}'_a(K(\mathcal{X}_a, \mathcal{X}_a, \mathcal{X}_a), \mathfrak{T}_1^2 v), \\ \mathfrak{C}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{C}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv), \\ \mathfrak{C}_a(J\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), Lv) \end{array} \right\}. \quad (4.11)$$

If $i = 1$, and

$$\inf_{L^{1-1} \in (0, \infty)} : \left\{ \begin{array}{l} \mathfrak{A}_a\left(\mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1 \omega\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), v\right) \geq \mathfrak{A}'_a\left(K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), \mathfrak{T}_1 v\right), \\ \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{B}_a\left(\mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1 \omega\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), t\right) \leq \mathfrak{B}'_a\left(K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), \mathfrak{T}_1 v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{C}_a\left(\mathcal{A}_1(\mathcal{X}_a) - \mathfrak{T}_1 \omega\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), t\right) \leq \mathfrak{C}'_a\left(K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), \mathfrak{T}_1 v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), v), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), v) \end{array} \right\} \quad (4.12)$$

and

$$\inf \left\{ L^{1-i} \in (0, \infty) : \begin{cases} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - J\mathcal{A}_1(\mathcal{X}_a), v) \leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v) \end{cases} \right\}. \quad (4.13)$$

Hence property [49] Condition : 1 holds.

According to the Condition : 2, [49] there is a alternative fixed point $\mathcal{A}_1$ of J in $\mathcal{S}$ then

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{A}_a \left(\frac{\mathcal{A}_1(\mathfrak{D}_i^n \mathcal{X}_a)}{\mathfrak{D}_i^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &= 1, \\ \lim_{n \rightarrow \infty} \mathfrak{B}_a \left(\frac{\mathcal{A}_1(\mathfrak{D}_i^n \mathcal{X}_a)}{\mathfrak{D}_i^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &= 0, \\ \lim_{n \rightarrow \infty} \mathfrak{C}_a \left(\frac{\mathcal{A}_1(\mathfrak{D}_i^n \mathcal{X}_a)}{\mathfrak{D}_i^n} - \mathcal{A}_1(\mathcal{X}_a), v \right) &= 0. \end{aligned}$$

For $\mathcal{A}_1$ is additive, taking $(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a)$ by $(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a)$.

Fixed point Condition : 3 of [49], $\mathcal{A}_1$ is the alternative fixed point of J in the set $\Delta = \{\mathcal{A}_1 \in \mathcal{S} : d(\mathcal{A}_1, \mathcal{A}) < \infty\}$, and $\mathcal{A}_1$ is a unique mapping then

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v), \quad \mathcal{X}_a \in \mathcal{L}, \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v), \quad \mathcal{X}_a \in \mathcal{L}, \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a(\mathcal{A}_1(\mathcal{X}_a), L^{1-i}v), \quad \mathcal{X}_a \in \mathcal{L}. \end{aligned} \right\}$$

By applying [49] Condition : 4

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a \left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L}v \right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a \left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L}v \right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a \left(\mathcal{A}_1(\mathcal{X}_a), \frac{L^{1-i}}{1-L}v \right). \end{aligned} \right\}$$

Hence proved. $\square$

Corollary 4.2 Let $\mathcal{A}_1$ be an approximately additive mapping from $(Z, \mathfrak{A}'_a, \mathfrak{B}'_a, \mathfrak{C}'_a)$ in a neutrosophic normed space and $(\mathcal{M}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$ be a neutrosophic Banach space satisfying the inequality

$$\begin{aligned} &\mathfrak{A}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) \\ &\geq \begin{cases} \mathfrak{A}'_a(\mathcal{S}, v), \\ \mathfrak{A}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{V}_a\|^\mathfrak{E} + \|\mathcal{W}_a\|^\mathfrak{E}), v), & \mathfrak{E} \neq 1, \\ \mathfrak{A}'_a(\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E}, v), & 3\mathfrak{E} \neq 1, \\ \mathfrak{A}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E} + (\|\mathcal{X}_a\|^{3\mathfrak{E}} + \|\mathcal{V}_a\|^{3\mathfrak{E}} + \|\mathcal{W}_a\|^{3\mathfrak{E}})\}, v), & 3\mathfrak{E} \neq 1, \end{cases} \end{aligned}$$

$$\begin{aligned}
& \mathfrak{B}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) \\
& \leq \begin{cases} \mathfrak{B}'_a(\mathcal{S}, v), \\ \mathfrak{B}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{V}_a\|^\mathfrak{E} + \|\mathcal{W}_a\|^\mathfrak{E}), v), & \mathfrak{E} \neq 1, \\ \mathfrak{B}'_a(\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E}, v), & 3\mathfrak{E} \neq 1, \\ \mathfrak{B}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E} + (\|\mathcal{X}_a\|^{3\mathfrak{E}} + \|\mathcal{V}_a\|^{3\mathfrak{E}} + \|\mathcal{W}_a\|^{3\mathfrak{E}})\}, v), & 3\mathfrak{E} \neq 1, \end{cases} \\
& \mathfrak{C}_a(\mathfrak{Z}(\mathcal{X}_a, \mathcal{V}_a, \mathcal{W}_a), v) \\
& \leq \begin{cases} \mathfrak{C}'_a(\mathcal{S}, v), \\ \mathfrak{C}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\mathfrak{E} + \|\mathcal{V}_a\|^\mathfrak{E} + \|\mathcal{W}_a\|^\mathfrak{E}), v), & \mathfrak{E} \neq 1, \\ \mathfrak{C}'_a(\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E}, v), & 3\mathfrak{E} \neq 1, \\ \mathfrak{C}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\mathfrak{E}\|\mathcal{V}_a\|^\mathfrak{E}\|\mathcal{W}_a\|^\mathfrak{E} + (\|\mathcal{X}_a\|^{3\mathfrak{E}} + \|\mathcal{V}_a\|^{3\mathfrak{E}} + \|\mathcal{W}_a\|^{3\mathfrak{E}})\}, v), & 3\mathfrak{E} \neq 1 \end{cases}
\end{aligned}$$

such that

$$\begin{aligned}
& \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \geq \begin{cases} \mathfrak{A}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1 - \mathfrak{T}_1}v\right), \\ \mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1}v\right), \\ \mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right), \\ \mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right)\right), \end{cases} \\
& \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \begin{cases} \mathfrak{B}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1 - \mathfrak{T}_1}v\right), \\ \mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1}v\right), \\ \mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right), \\ \mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right)\right), \end{cases} \quad (4.14) \\
& \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) \leq \begin{cases} \mathfrak{C}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1 - \mathfrak{T}_1}v\right), \\ \mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1}v\right), \\ \mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right), \\ \mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}} - \mathfrak{T}_1}v\right)\right). \end{cases}
\end{aligned}$$

Proof Let

$$\begin{aligned}
 & \mathfrak{A}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}_i^{\mathcal{L}_1} v) \\
 &= \begin{cases} \mathfrak{A}'_a(\mathcal{S}, \mathfrak{D}_i^k v), \\ \mathfrak{A}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon + \|\mathcal{V}_a\|^\epsilon + \|\mathcal{W}_a\|^\epsilon), \mathfrak{D}_i^{\mathcal{L}_1 - \epsilon} v), \\ \mathfrak{A}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v), \\ \mathfrak{A}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon + (\|\mathcal{X}_a\|^{3\epsilon} + \|\mathcal{V}_a\|^{3\epsilon} + \|\mathcal{W}_a\|^{3\epsilon})\}, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v) \end{cases} \\
 &= \begin{cases} \rightarrow 1 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 1 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 1 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 1 & \text{as } \mathcal{L}_1 \rightarrow \infty \end{cases} \\
 & \mathfrak{B}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}_i^{\mathcal{L}_1} v) \\
 &= \begin{cases} \mathfrak{B}'_a(\mathcal{S}, \mathfrak{D}_i^{\mathcal{L}_1} v), \\ \mathfrak{B}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon + \|\mathcal{V}_a\|^\epsilon + \|\mathcal{W}_a\|^\epsilon), \mathfrak{D}_i^{\mathcal{L}_1 - \epsilon} v), \\ \mathfrak{B}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v), \\ \mathfrak{B}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon + (\|\mathcal{X}_a\|^{3\epsilon} + \|\mathcal{V}_a\|^{3\epsilon} + \|\mathcal{W}_a\|^{3\epsilon})\}, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v) \end{cases} \\
 &= \begin{cases} \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty \end{cases} \\
 & \mathfrak{C}'_a(K(\mathfrak{D}_i^n \mathcal{X}_a, \mathfrak{D}_i^n \mathcal{V}_a, \mathfrak{D}_i^n \mathcal{W}_a), \mathfrak{D}_i^{\mathcal{L}_1} v) \\
 &= \begin{cases} \mathfrak{C}'_a(\mathcal{S}, \mathfrak{D}_i^{\mathcal{L}_1} v), \\ \mathfrak{C}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon + \|\mathcal{V}_a\|^\epsilon + \|\mathcal{W}_a\|^\epsilon), \mathfrak{D}_i^{\mathcal{L}_1 - \epsilon} v), \\ \mathfrak{C}'_a(\mathcal{S}(\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v), \\ \mathfrak{C}'_a(\mathcal{S}\{\|\mathcal{X}_a\|^\epsilon \|\mathcal{V}_a\|^\epsilon \|\mathcal{W}_a\|^\epsilon + (\|\mathcal{X}_a\|^{3\epsilon} + \|\mathcal{V}_a\|^{3\epsilon} + \|\mathcal{W}_a\|^{3\epsilon})\}, \mathfrak{D}_i^{\mathcal{L}_1 - 3\epsilon} v) \end{cases} \\
 &= \begin{cases} \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty, \\ \rightarrow 0 & \text{as } \mathcal{L}_1 \rightarrow \infty. \end{cases}
 \end{aligned}$$

Here (4.1) exists, then

$$\begin{aligned} \mathfrak{A}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), v\right) &= \left\{ \begin{aligned} &\mathfrak{A}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, v\right), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, v\right), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, v\right), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), v\right), \end{aligned} \right. \\ \mathfrak{B}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, v, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), v\right) &= \left\{ \begin{aligned} &\mathfrak{B}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, v\right), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, v\right), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, v\right), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), v\right), \end{aligned} \right. \\ \mathfrak{C}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}, \frac{\mathcal{X}_a}{\mathfrak{T}_1}\right), v\right) &= \left\{ \begin{aligned} &\mathfrak{C}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, v\right), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, v\right), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, v\right), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), v\right). \end{aligned} \right. \end{aligned}$$

From (4.5),

$$\begin{aligned}\mathfrak{A}'_a\left(\frac{\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \left\{ \begin{aligned} &\mathfrak{A}'_a(\mathcal{S}, \mathfrak{D}_i v), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \mathfrak{D}_i^{1-\mathfrak{E}} v\right), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \mathfrak{D}_i^{1-3\mathfrak{E}} v\right), \\ &\mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), \mathfrak{D}_i^{1-3\mathfrak{E}} v\right), \end{aligned} \right. \\ \mathfrak{B}'_a\left(\frac{\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \left\{ \begin{aligned} &\mathfrak{B}'_a(\mathcal{S}, \mathfrak{D}_i v), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \mathfrak{D}_i^{1-\mathfrak{E}} v\right), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \mathfrak{D}_i^{1-3\mathfrak{E}} v\right), \\ &\mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), \mathfrak{D}_i^{1-3\mathfrak{E}} v\right), \end{aligned} \right. \\ \mathfrak{C}'_a\left(\frac{\mathcal{A}_1(\mathfrak{D}_i\mathcal{X}_a)}{\mathfrak{D}_i}, v\right) &= \left\{ \begin{aligned} &\mathfrak{B}'_a(\mathcal{S}, \mathfrak{D}_i v), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \mathfrak{D}_i^{1-\mathfrak{E}} v\right), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}, \mathfrak{D}_i^{1-3\mathfrak{E}} v\right), \\ &\mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathcal{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1} \left(\frac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}} + \frac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right), \mathfrak{D}_i^{1-3\mathfrak{E}} v\right). \end{aligned} \right.\end{aligned}$$

Hence (4.6), we have

L	$\mathfrak{E}, i = 0$	L	$\mathfrak{E}, i = 1$
1. $\mathfrak{T}_1$	0	$\mathfrak{T}_1^{-1}$	0
2. $\mathfrak{T}_1^{1-\mathfrak{E}}$	$\mathfrak{E} < 1$	$\mathfrak{T}_1^{\mathfrak{E}-1}$	$\mathfrak{E} > 1$
3. $\mathfrak{T}_1^{1-3\mathfrak{E}}$	$3\mathfrak{E} < 1$	$\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} > 1$
4. $\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} < 1$	$\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} > 1$.

Method: 1 For $i = 0$,

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1} v\right) = \mathfrak{A}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1-\mathfrak{T}_1} v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1} v\right) = \mathfrak{B}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1-\mathfrak{T}_1} v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1} v\right) = \mathfrak{C}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{1-\mathfrak{T}_1} v\right). \end{aligned} \right\}$$

Method: 1 For $i = 1$,

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1 - (\mathfrak{T}_1)^{-1}} v\right) = \mathfrak{A}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1} v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1 - (\mathfrak{T}_1)^{-1}} v\right) = \mathfrak{B}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1} v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1 - (\mathfrak{T}_1)^{-1}} v\right) = \mathfrak{C}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1} v\right). \end{aligned} \right\}$$

Method: 2 For $i = 0$,

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1 - (\mathfrak{T}_1^{1-\mathfrak{E}})} v\right) \\ &= \mathfrak{A}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1^k} v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1 - (\mathfrak{T}_1^{1-\mathfrak{E}})} v\right) \\ &= \mathfrak{B}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1} v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1 - (\mathfrak{T}_1^{1-\mathfrak{E}})} v\right) \\ &= \mathfrak{C}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^\mathfrak{E} - \mathfrak{T}_1} v\right). \end{aligned} \right\}$$

Method: 2 For $i = 1$,

$$\left. \begin{aligned} \mathfrak{A}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\geq \mathfrak{A}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1 - (\mathfrak{T}_1^{A-1})} v\right) \\ &= \mathfrak{A}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^k - \mathfrak{T}_1^a} v\right), \\ \mathfrak{B}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{B}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1 - (\mathfrak{T}_1^{A-1})} v\right) \\ &= \mathfrak{B}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - \mathfrak{T}_1^a} v\right), \\ \mathfrak{C}_a(\mathcal{A}_1(\mathcal{X}_a) - \mathcal{A}_1(\mathcal{X}_a), v) &\leq \mathfrak{C}'_a\left(\mathcal{A}_1(\mathcal{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1 - (\mathfrak{T}_1^{A-1})} v\right) \\ &= \mathfrak{C}'_a\left(\frac{\mathcal{S} \|\mathcal{X}_a\|^\mathfrak{E}}{\mathfrak{T}_1} \frac{3}{|\mathfrak{T}_1|^\mathfrak{E}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - \mathfrak{T}_1^a} v\right). \end{aligned} \right\}$$

□

5 Conclusion

This paper has presented a novel approach for analyzing the stability of the Euler-Lagrange additive FE within neutrosophic normed spaces, addressing a critical need in the realm of uncertainty modeling and functional analysis. The investigation delves into the existence of solutions for this equation and explores its Ulam-Hyers stability within neutrosophic normed spaces. The results are obtained by applying two distinct approaches: direct and

fixed point techniques. The findings presented in this article establish the relationship between four distinct areas of research: FEs, neutrosophic normed spaces, Ulam-Hyers stability, and fixed point theory. The stability analysis of this equation in neutrosophic normed spaces is unique, given the absence of prior research on the stability of equations employing neutrosophic concepts. This paper contributes to the broader advancement of neutrosophic mathematics by offering new perspectives and methodologies for analyzing functional equations in nonclassical settings. Our work opens up avenues for further research and exploration, inviting interdisciplinary collaboration and innovation in the fields of mathematics, uncertainty modeling, and system analysis.

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Author contributions

A.A., P.A., K.J., S.A. and N.M. wrote the main manuscript text. All authors reviewed the manuscript.

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors declare no competing interests.

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