



Bell wavelet-based numerical algorithm for fractional-order $(1+1)$ -dimensional telegraph equations involving derivative in Caputo sense

Pooja Yadav¹ · Shah Jahan¹

Received: 7 October 2024 / Revised: 12 November 2024 / Accepted: 9 December 2024 / Published online: 18 January 2025
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2024, corrected publication 2025

Abstract

This study presents an efficient numerical approach for solving fractional-order $(1+1)$ -dimensional telegraph equations using a collocation method based on Bell wavelets. The method begins by exploring the fundamental properties of Bell polynomials and their associated wavelets. Fractional-order operational matrices of integrals (OMI) are developed using block-pulse functions, and the Bell wavelet OMI is then combined with collocation points to transform the telegraph equation into a system of algebraic equations. Newton's iterative technique is used to solve these equations. To validate the effectiveness and applicability of the proposed Bell wavelet method, several test problems are analyzed, with detailed error analysis included. Comparative results between the Bell wavelet approximate solutions and those from other established methods in the existing literature are presented through comprehensive figures and tables. This comparison highlights the improved accuracy and efficiency of the Bell wavelet method, demonstrating its potential as a reliable tool for solving fractional telegraph equations.

Keywords Bell polynomials · Bell wavelets · Fractional telegraph equation · Error analysis · Collocation points · Operational matrix

Mathematics Subject Classification 42C40 · 26A33 · 65M70 · 35R11 · 65T60

1 Introduction

Oliver Heaviside formulated the telegraph equation during the 1880s. It is a linear, second-order hyperbolic partial differential equation that characterizes how current and voltage behave along an electrical transmission line in relation to time and distance. This model shows how wave patterns develop along the wire and how the wire can reflect electromagnetic radiation [1]. Numerous domains, including engineering and biological sciences, use the telegraph equation to simulate reaction–diffusion processes, research wave phenomena, and track electrical impulses coursing through cable transmission lines. In the physical, biological, and chemical sciences, a variety of events are governed by the Telegraph Equation. Examples include population dynamics [2], parallel flows in Maxwell fluids [4], acoustic waves in porous media [3],

wave propagation in cable transmissions [5], hyperbolic heat transfer [7], and pulsatile blood flow in arteries [6]. Due to numerous applications of telegraph equations in signal analysis and wave propagation, we are motivated to solve these equations by leveraging the properties of fractional calculus alongside Bell wavelets.

Fractional calculus is a branch of mathematics that explores derivatives and integrals of non-integer orders, creating opportunities for collaboration among engineers, mathematicians, biologists, and physicists [39–43]. Fractional derivatives are essential for describing various physical processes, as they extend traditional derivatives to provide a more precise and efficient modeling tool. Their non-local nature captures complex behaviors in ways that integer-order derivatives cannot. Fractional calculus finds extensive applications in fields such as fluid mechanics [8], chemical engineering [10], viscoelasticity [9], and signal processing [11]. However, solving fractional differential equations with exact solutions poses significant challenges due to the fractional nature of their kernels. Consequently, extensive research has been dedicated to developing numerical

✉ Shah Jahan
shahjahan@cuh.ac.in

¹ Department of Mathematics, Central University of Haryana,
Mohindergarh 123031, India

approaches to handle fractional differential equations [37, 38]. These approaches encompass numerous techniques, including the Adomian decomposition method [32], Laplace transform method [33], Sinc-Legendre method [29], Legendre operational matrix [28], Fibonacci wavelet [30], and others [31].

Because of the advantageous properties associated with fractional calculus, we explore the time-fractional telegraph equation in this paper as

$$\begin{aligned} & \frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + a_1 \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + a_2 \theta(\lambda, \delta) \\ & = a_3 \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \end{aligned} \quad (1.1)$$

with initial conditions (ICs)

$$\theta(\lambda, 0) = \mu_1(\lambda), \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = \mu_2(\lambda), \quad 0 \leq \lambda \leq 1, \quad (1.2)$$

and Dirichlet's boundary conditions (DBC's)

$$\theta(0, \delta) = v_0(\delta), \quad \theta(1, \delta) = v_1(\delta), \quad 0 < \delta \leq 1, \quad (1.3)$$

where a_1, a_2, a_3 are arbitrary constants, and $f(\lambda, \delta)$ is a continuous real valued function. $1 < \alpha \leq 2$ represents the fractional derivative parameter. At $\alpha=2$, (1.1) becomes the classical telegraph equation.

For solving fractional partial differential equations, various numerical methodologies have been utilized, including the spectral method [12], finite element, and difference method [13, 14]. Many investigations have explored the computational resolution of fractional differential equations and their practical implications [15, 16].

In recent years, the fractional telegraph equation (1.1) has gained significant attention in engineering and physical sciences, leading to various research advancements. Several methods have been developed and successfully applied to this area, including the Adomian decomposition method [17], separation of variables [18], finite difference method [36], Sinc-Legendre collocation method [19], differential transform method [20], Crank–Nicolson finite difference method [21], Fourier and Laplace transform methods [22], cubic spline [34], and homotopy analysis and homotopy perturbation methods [23, 24]. Among these approaches, wavelet-based numerical techniques have emerged as a promising alternative for solving fractional telegraph equations.

Wavelet-based numerical methods have become popular in numerical analysis due to their simplicity, computational efficiency, and fast convergence [35]. Recently, various wavelets, such as Bernoulli, CAS, Chebyshev, Legendre, Haar, Gegenbauer, and Taylor wavelets, have been used to solve problems in fractional calculus. These wavelets employ

operational matrices for fractional integration, often using block-pulse functions to simplify computations. A newer addition to wavelet families is the Bell wavelet, which is generated using Bell polynomials.

This study introduces a novel, efficient computational method for solving fractional-order telegraph equations with DBC's by combining block-pulse functions (BPFs) and Bell wavelets. This method was chosen for its ability to overcome the computational and accuracy challenges of fractional-order problems, which are common in systems with memory-dependent behavior. By integrating Bell wavelets as the foundational basis with the operational matrix and collocation methods, the approach leverages the OMI to convert complex differential equations into simpler algebraic equations, solvable with Newton's iterative technique [25]. This combination enhances computational efficiency and flexibility, as Bell wavelets enable precise local approximations, while the operational matrix method reduces complexity. Through various numerical examples, the method demonstrates high accuracy and reliability, proving useful for a broad range of fractional-order differential equations. No significant limitations of this strategy are immediately apparent. However, the approach is most effective within a limited domain and may not be as applicable in broader contexts. Computing the operational matrix and performing integrations with the Bell wavelet method can become resource-intensive, particularly for high-dimensional problems or cases that require many terms in the series expansion.

The structure of the paper is as follows: Sect. 2 provides the essential preliminary knowledge of fractional calculus, Bell polynomials, Bell wavelets, and their corresponding OMI. In Sect. 3, we introduce a fractional OMI, utilizing Bell wavelets as the basis. In Sects. 4 and 5, we discuss the description of methods and convergence analysis. Section 6 is dedicated to presenting our numerical simulations and discussion, showcasing the efficacy of the presented methodology through illustrative examples. Lastly, Sect. 7 gives a brief conclusion and future directions.

2 Bell wavelet and operational matrices of integration

Here, some definitions from fractional calculus, Bell polynomials, and Bell wavelets are given.

Fractional calculus

Definition 2.1 For any function $\theta(\lambda) \in C_\omega$, $\omega \geq -1$, the R-L fractional integration operator I^a ($a > 0$) is given by:

$$I^a \theta(\lambda) = \begin{cases} \theta(\lambda), & \lambda = 0, \\ \frac{1}{\Gamma(a)} \int_0^\lambda (\lambda - x)^{a-1} \theta(x) dx, & a > 0. \end{cases} \quad (2.1)$$

Properties of the R-L fractional integration operator for $a, b > 0$:

- $I^a I^b \theta(\lambda) = I^{a+b} \theta(\lambda),$
- $I^a I^b \theta(\lambda) = I^{a+b} \theta(\lambda),$
- $I^a \lambda^v = \frac{\Gamma(1+v)}{\Gamma(1+a+v)} \lambda^{a+v}, \quad v > -1.$

Definition 2.2 For any function $\theta(\lambda) \in C_{-1}^n$, the Liouville–Caputo fractional derivative operator $\mathcal{D}^a$ is given by

$$\mathcal{D}^a \theta(\lambda) = \begin{cases} \theta^{(n)}(\lambda), & a = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-a)} \int_0^\lambda \frac{\theta^{(n)}(x)}{(\lambda-x)^{a+1-n}} dx, & n-1 < a < n, n \in \mathbb{N}. \end{cases} \quad (2.2)$$

From (2.1) and (2.2), we get

- $\mathcal{D}^a I^a \theta(\lambda) = \theta(\lambda),$
- $\mathcal{D}^b I^a \theta(\lambda) = I^{a-b} \theta(\lambda), \quad a > b,$
- $I^a \mathcal{D}^a \theta(\lambda) = \theta(\lambda) - \sum_{k=0}^{n-1} \theta^{(k)}(0^+) \frac{\lambda^k}{k!}, \quad \lambda > 0, \quad n-1 < a \leq n.$

Bell wavelets

Bell polynomials $\mathcal{B}_n(\lambda)$ of order n are defined by [27]

$$\mathcal{B}_n(\lambda) = \sum_{k=0}^n S(n, k) \lambda^k,$$

here $S(n, k)$ represents the second-kind Stirling number and defined as:

$$S(n, k) = \sum_{i=0}^k \frac{(-1)^i}{k!} \binom{k}{i} (k-i)^n.$$

Using the Bell polynomials properties, the vector $\mathcal{B}(\lambda) = [\mathcal{B}_0(\lambda), \dots, \mathcal{B}_n(\lambda)]$, consisting the Bell polynomials $\mathcal{B}_j(\lambda)$, $j = 0, 1, \dots, n$, is represented by

$$\mathcal{B}(\lambda) = \mathbf{S} \mathbf{Y}(\lambda), \quad (2.3)$$

where

$$\mathbf{Y}(\lambda) = [1, \lambda, \dots, \lambda^n]^T,$$

and

$$\mathbf{S} = \begin{bmatrix} S(0, 0) & 0 & \cdots & 0 \\ S(1, 0) & S(1, 1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S(n, 0) & S(n, 1) & \cdots & S(n, n) \end{bmatrix}.$$

Since the above matrix is nonsingular as $\mathbf{S}$ is a lower triangular matrix with nonzero diagonal elements, $\mathbf{S}^{-1}$ exists. So, equation (2.3) becomes

$$\mathbf{Y}(\lambda) = \mathbf{S}^{-1} \mathcal{B}(\lambda).$$

The first six Bell polynomials are

$$\begin{aligned} \mathcal{B}_0(\lambda) &= 1, \\ \mathcal{B}_1(\lambda) &= \lambda, \\ \mathcal{B}_2(\lambda) &= \lambda^2 + \lambda, \\ \mathcal{B}_3(\lambda) &= \lambda^3 + 3\lambda^2 + \lambda, \\ \mathcal{B}_4(\lambda) &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda, \\ \mathcal{B}_5(\lambda) &= \lambda^5 + 10\lambda^4 + 25\lambda^3 + 15\lambda^2 + \lambda, \\ \mathcal{B}_6(\lambda) &= \lambda^6 + 15\lambda^5 + 65\lambda^4 + 90\lambda^3 + 31\lambda^2 + \lambda. \end{aligned}$$

The Bell wavelets $\psi_{mn}(\lambda) = \psi(k, m, n, \lambda)$ have four arguments: $k \in \mathbb{Z}^+$, $m = 1, \dots, 2^{k-1}$, the λ is the normalized time, and the degree of the Bell polynomials is given by n . On the interval $[0, 1)$, Bell wavelets are given by

$$\psi_{mn}(\lambda) = \begin{cases} 2^{\frac{k-1}{2}} \mathcal{B}_n(2^{k-1}\lambda - m + 1), & \frac{m-1}{2^{k-1}} \leq \lambda < \frac{m}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

The Bell wavelets basis for $k = 2$, $M = 3$ is

$$\psi_{1,0}(\lambda) = \begin{cases} \sqrt{2}, & 0 \leq \lambda < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{1,1}(\lambda) = \begin{cases} 2\sqrt{2}\lambda, & 0 \leq \lambda < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{1,2}(\lambda) = \begin{cases} 4\sqrt{2}\lambda^2 + 2\sqrt{2}\lambda, & 0 \leq \lambda < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{2,0}(\lambda) = \begin{cases} \sqrt{2}, & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{2,1}(\lambda) = \begin{cases} 2\sqrt{2}\lambda - \sqrt{2}, & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_{2,2}(\lambda) = \begin{cases} 4\sqrt{2}\lambda^2 - 2\sqrt{2}\lambda, & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix will be for $k = 2$, $M = 3$

$$\psi_{6 \times 6} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 0.2357 & 0.7071 & 1.1785 & 0 & 0 & 0 \\ 0.2750 & 1.0607 & 2.1606 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0.2357 & 0.7071 & 1.1785 \\ 0 & 0 & 0 & 0.2750 & 1.0607 & 2.1606 \end{bmatrix}.$$

The integrals that will be used to solve fractional telegraph equations are as follows:

$$q_{m,n}(\lambda) = \int_0^\lambda \psi_{mn}(z) dz, \quad p_{m,n}(\lambda) = \int_0^\lambda p_{mn}(z) dz, \quad (2.5)$$

and the elements of the matrices for (2.5) are given by

$$(q_{m,n})_{2^{k-1} \times M} = \begin{pmatrix} q_{1,0}(\lambda) & q_{1,1}(\lambda) & \cdots & q_{1,M-1}(\lambda) \\ q_{2,0}(\lambda) & q_{2,1}(\lambda) & \cdots & q_{2,M-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ q_{2^{k-1},0} & q_{2^{k-1},1} & \cdots & q_{2^{k-1},M-1}(\lambda) \end{pmatrix}_{2^{k-1} \times M},$$

and

$$(p_{m,n})_{2^{k-1} \times M} = \begin{pmatrix} p_{1,0}(\lambda) & p_{1,1}(\lambda) & \cdots & p_{1,M-1}(\lambda) \\ p_{2,0}(\lambda) & p_{2,1}(\lambda) & \cdots & p_{2,M-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ p_{2^{k-1},0} & p_{2^{k-1},1} & \cdots & p_{2^{k-1},M-1}(\lambda) \end{pmatrix}_{2^{k-1} \times M}.$$

By taking $k = 2$, $M = 3$

$$q_{1,0}(\lambda) = \begin{cases} \sqrt{2}\lambda, & 0 \leq \lambda < \frac{1}{2}, \\ \frac{\sqrt{2}}{2}, & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$q_{1,1}(\lambda) = \begin{cases} \sqrt{2}\lambda^2, & 0 \leq \lambda < \frac{1}{2}, \\ \frac{\sqrt{2}}{4}, & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$q_{1,2}(\lambda) = \begin{cases} 2\sqrt{2}\left(\frac{2}{3}\lambda^3 + \frac{\lambda^2}{2}\right), & 0 \leq \lambda < \frac{1}{2}, \\ 2\sqrt{2}\left(\frac{5}{24}\right), & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$q_{2,0}(\lambda) = \begin{cases} \sqrt{2}\left(\lambda - \frac{1}{2}\right), & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$q_{2,1}(\lambda) = \begin{cases} \sqrt{2}\left(\lambda^2 - \lambda + \frac{1}{4}\right), & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$q_{2,2}(\lambda) = \begin{cases} 2\sqrt{2}\left(\frac{2}{3}\lambda^2 - \frac{\lambda^2}{2} - \frac{5}{24}\right), & \frac{1}{2} \leq \lambda < 1., \\ 0, & \text{otherwise.} \end{cases}$$

and

$$p_{1,0}(\lambda) = \begin{cases} \frac{1}{\sqrt{2}}\lambda^2, & 0 \leq \lambda < \frac{1}{2}, \\ \sqrt{2}\left(\frac{\lambda}{2} - \frac{1}{8}\right), & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$p_{1,1}(\lambda) = \begin{cases} \frac{\sqrt{2}}{3}\lambda^3, & 0 \leq \lambda < \frac{1}{2}, \\ \sqrt{2}\left(\frac{\lambda}{4} - \frac{1}{12}\right), & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$p_{1,2}(\lambda) = \begin{cases} 2\sqrt{2}\left(\frac{1}{6}\lambda^4 + \frac{\lambda^3}{6}\right), & 0 \leq \lambda < \frac{1}{2}, \\ 2\sqrt{2}\left(\frac{5}{24}\lambda - \frac{7}{96}\right), & \frac{1}{2} \leq \lambda < 1. \end{cases}$$

$$p_{2,0}(\lambda) = \begin{cases} \sqrt{2}\left(\frac{\lambda^2}{2} - \frac{\lambda}{2} + \frac{1}{8}\right), & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{2,1}(\lambda) = \begin{cases} \sqrt{2}\left(\frac{\lambda^3}{3} - \frac{\lambda^2}{2} + \frac{\lambda}{4} - \frac{1}{24}\right), & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{2,2}(\lambda) = \begin{cases} 2\sqrt{2}\left(\frac{1}{6}\lambda^4 - \frac{\lambda^3}{6} - \frac{5}{24}\lambda - \frac{9}{96}\right), & \frac{1}{2} \leq \lambda < 1, \\ 0, & \text{otherwise.} \end{cases}$$

A function $\theta(\lambda) \in L^2[0, 1]$ could be represented in the form of Bell wavelets as:

$$\theta(\lambda) = \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}^+} c_{mn} \psi_{mn}(\lambda).$$

By truncating above equation, we get

$$\theta(\lambda) \simeq \sum_{m=1}^{2^{k-1}} \sum_{n=0}^{M-1} c_{mn} \psi_{mn}(\lambda), \quad (2.6)$$

where

$$c_{m,n} = \int_0^1 \theta(\lambda) \psi_{m,n}(\lambda) d\lambda,$$

are the coefficients of Bell wavelet. The equation (2.6) can be expressed as a matrix:

$$\theta(\lambda) = C^T \Psi(\lambda), \quad (2.7)$$

here we define the row vector C as:

$$C = [c_{1,0}, \dots, c_{1,M-2}, c_{1,M-1}, c_{2,0}, \dots, c_{2,M-2}, c_{2,M-1}, \dots, c_{2^{k-1},0}, c_{2^{k-1},1}, \dots, c_{2^{k-1},M-1}]^T. \quad (2.8)$$

The matrix $\Psi(\lambda)$ in (2.7) is $1 \times 2^{k-1}M$ -order Bell wavelet matrix is as

$$\Psi(\lambda) = [\psi_{1,0}, \dots, \psi_{1,M-2}, \psi_{1,M-1}, \psi_{2,0}, \dots, \psi_{2,M-2}, \psi_{2,M-1}, \dots, \psi_{2^{k-1},0}, \psi_{2^{k-1},1}, \dots, \psi_{2^{k-1},M-1}]^T. \quad (2.9)$$

Finally, taking the collocation points:

$$\lambda_\ell = \frac{2\ell - 1}{2^k M}, \quad \ell = 1, 2, \dots, 2^{k-1}M. \quad (2.10)$$

From (2.4), the Bell wavelets in two dimensions can be defined as:

$$\psi_{m_1, n_1, m_2, n_2}(\lambda, \delta) = \begin{cases} 2^{(k_1-1)/2} 2^{(k_2-1)/2} \mathcal{B}_{n_1}(2^{k_1-1}\lambda - m_1 + 1) \mathcal{B}_{n_2}(2^{k_2-1}\delta - m_2 + 1), \\ \quad \frac{m_1-1}{2^{k_1-1}} \leq \lambda < \frac{m_1}{2^{k_1-1}}, \frac{m_2-1}{2^{k_2-1}} \leq \delta < \frac{m_2}{2^{k_2-1}} \\ 0, & \text{otherwise,} \end{cases} \quad (2.11)$$

where m_1, n_1, m_2, n_2, k_1 and k_2 have the usual meanings and $\psi_{m_1, n_1, m_2, n_2}(\lambda, \delta)$ constitute a basis for $L^2([0, 1) \times [0, 1))$.

So, equation (2.11) can be expressed by any arbitrary function $\theta(\lambda, \delta) \in L^2([0, 1) \times [0, 1))$ as

$$\begin{aligned} \theta(\lambda, \delta) &= \sum_{m_1=1}^{2^{k_1-1}M_1-1} \sum_{n_1=0}^{2^{k_1-1}M_1-1} \sum_{m_2=1}^{2^{k_2-1}M_2-1} \sum_{n_2=0}^{2^{k_2-1}M_2-1} c_{m_1, n_1, m_2, n_2} \psi_{m_1, n_1}(\lambda) \psi_{m_2, n_2}(\delta), \\ &= \psi^T(\lambda) C \psi(\delta); \end{aligned} \quad (2.12)$$

here, C is the matrix of order $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$ composed of Bell wavelet coefficients

$$\begin{aligned} c_{m_1, n_1, m_2, n_2} &= \int_0^1 \int_0^1 \theta(\lambda, \delta) \psi_{m_1, n_1}(\lambda) \psi_{m_2, n_2}(\delta) d\lambda d\delta, \\ 1 \leq m_1 \leq 2^{k_1-1}M_1, 0 \leq n_1 \leq M_1 - 1, \\ 1 \leq m_2 \leq 2^{k_2-1}M_2, 0 \leq n_2 \leq M_2 - 1. \end{aligned} \quad (2.13)$$

Finally, taking collocation point:

$$\begin{aligned} \lambda_{\ell_1} &= \frac{2\ell_1 - 1}{2^{k_1}M_1}, \quad \ell_1 = 1, 2, \dots, 2^{k_1-1}M_1. \\ \delta_{\ell_2} &= \frac{2\ell_2 - 1}{2^{k_2}M_2}, \quad \ell_2 = 1, 2, \dots, 2^{k_2-1}M_2. \end{aligned} \quad (2.14)$$

3 Fractional OMI of bell wavelet

On $[0, 1)$ interval, the BPFs are given by

$$b_v(\lambda) = \begin{cases} 1 & vq \leq \lambda < (v+1)q, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

where $q = \frac{1}{N}$ and for $N \in \mathbb{Z}^+$, $v = 0, 1, \dots, N-1$.

The function $\theta(\lambda) \in L^2[0, 1)$ is estimated via BPFs.

$$\theta(\lambda) \simeq \theta_N(\lambda) = \sum_{v=1}^{N-1} b_v g_v(\lambda) = B^T G_N, \quad (3.2)$$

where $G_N = [G_0, G_1, G_2, \dots, G_N]^T$ and $B = [b_0, b_1, b_2, \dots, b_N - 1]$. Integrating the vector $G_N(\lambda)$, we get

$$\int_0^\lambda G_N(y) dy \simeq \Delta G_N(\lambda), \quad (3.3)$$

so the OMI for BPFs is defined as follow:

$$\Delta = \frac{q}{2} \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (3.4)$$

Then, by using BPFs we describe the OMI of fractional-order H^α as

$$(I^\alpha G_N)(\lambda) \simeq H^\alpha G_N(\lambda), \quad (3.5)$$

where

$$H^\alpha = \frac{1}{N^\alpha \Gamma(\alpha + 2)} \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_{N-1} \\ 0 & 1 & \lambda_1 & \lambda_2 & \dots & \lambda_{N-2} \\ 0 & 0 & 1 & \lambda_1 & \dots & \lambda_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \lambda_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad (3.6)$$

where the λ_v 's in (3.6) have been defined as:

$$\lambda_v = (v + 1)^{\alpha+1} - 2v^{\alpha+1} + (v - 1)^{\alpha+1}. \quad (3.7)$$

3.1 OMI of bell wavelet

Here, using BPFs, fractional-order integration matrices for Bell wavelets are created by integrating equation (2.9) as follows:

$$\int_0^\lambda \Psi(y) dy \approx O \Psi(\lambda), \quad (3.8)$$

where O represents the OMI for $2^{k-1}M \times 2^{k-1}M$ order of Bell wavelet. It is worth mentioning that BPFs (3.1) are used to represent the Bell wavelets (2.9) as

$$\psi(\lambda) = \psi_{m,n} G_N(\lambda). \quad (3.9)$$

To get the OMI for order α of Bell wavelet, we define

$$I^\alpha \psi(\lambda) = O_{m,n}^\alpha \psi(\lambda), \quad (3.10)$$

where the $O_{m,n}^\alpha$ matrix defines the Bell wavelet's OMI of fractional order; by taking the relationships (3.5), (3.9), and (3.10), we get

$$\begin{aligned} (I^\alpha \psi)(\lambda) &\approx (I^\alpha \psi_{m,n} G_N)(\lambda) = \psi_{m,n} (I^\alpha G_N)(\lambda) \\ &\approx \psi_{m,n} H^\alpha G_N(\lambda). \end{aligned} \quad (3.11)$$

From (3.10) and (3.11), we obtain:

$$O_{m,n}^\alpha \psi(\lambda) = O_{m,n}^\alpha \psi_{m,n} G_N(\lambda) = \psi_{m,n} H^\alpha G_N(\lambda), \quad (3.12)$$

which yields in the necessary OMI of general order for the Bell wavelets:

$$O_{m,n}^\alpha = \psi_{m,n} H^\alpha [\psi_{m,n}]^{-1}. \quad (3.13)$$

For instance, at $\alpha = 0.6$, $k = 2$, $M = 3$, the corresponding fractional OMI $O_{6 \times 6}^{0.6}$ corresponding with the Bell wavelets is given by:

$$O_{6 \times 6}^{0.6} = \begin{pmatrix} 0.0892 & 1.2664 & -0.3166 & 0.6498 & -0.7848 & 0.2653 \\ 0.0059 & -0.1658 & 0.3166 & 0.3712 & -0.5411 & 0.1895 \\ 0.0222 & -0.7797 & 0.7927 & 0.6306 & -0.9550 & 0.3368 \\ 0 & 0 & 0 & 0.0892 & 1.2664 & -0.3166 \\ 0 & 0 & 0 & 0.0059 & -0.1658 & 0.3166 \\ 0 & 0 & 0 & 0.0222 & -0.7797 & 0.7927 \end{pmatrix}. \quad (3.14)$$

4 Description of method

Here, the time-fractional telegraph equation is solved using Bell wavelet method having both ICs and DBCs. Consider the following equation

$$\begin{aligned} \frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + \theta(\lambda, \delta) \\ = \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \end{aligned} \quad (4.1)$$

with ICs

$$\theta(\lambda, 0) = \mu_1(\lambda), \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = \mu_2(\lambda), \quad 0 \leq \lambda \leq 1, \quad (4.2)$$

and DBCs

$$\theta(0, \delta) = \nu_0(\delta), \quad \theta(1, \delta) = \nu_1(\delta), \quad 0 < \delta \leq 1. \quad (4.3)$$

Firstly, by approximating the highest partial derivative by Bell wavelet such that

$$\frac{\partial^4 \theta(\lambda, \delta)}{\partial \lambda^2 \partial \delta^2} \approx \Psi^T(\lambda) C \Psi(\delta), \quad (4.4)$$

here C is the undefined Bell wavelet coefficient vector provided in (2.8). After integrating (4.4) two times w.r.t δ , we have

$$\frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} \approx \Psi^T(\lambda) C \left(O^2 \Psi(\delta) \right) + \delta \mu_2''(\lambda) + \mu_1''(\lambda). \quad (4.5)$$

Now, by integrating (4.5) twice w.r.t λ we obtain

$$\begin{aligned} \theta(\lambda, \delta) &\approx \left(O^2 \Psi(\lambda) \right)^T C \left(O^2 \Psi(\delta) \right) \\ &\quad + \delta \left(\mu_2(\lambda) - \mu_2(0) - \lambda \mu_2'(0) \right) \\ &\quad + \left(\mu_1(\lambda) - \mu_1(0) - \lambda \mu_1'(0) \right) \\ &\quad + \lambda \left. \frac{\partial \theta(\lambda, \delta)}{\partial \lambda} \right|_{\lambda=0} + \theta(0, \delta). \end{aligned} \quad (4.6)$$

By taking $\lambda = 1$ in (4.6) and applying conditions (4.3), we get

$$\begin{aligned} \theta(1, \delta) \approx & \left(O^2 \Psi(1) \right)^T C \left(O^2 \Psi(\delta) \right) \\ & + \delta \left(\mu_2(1) - \mu_2(0) - \mu'_2(0) \right) \\ & + \left(\mu_1(1) - \mu_1(0) - \mu'_1(0) \right) \\ & + \left. \frac{\partial \theta(\lambda, \delta)}{\partial \lambda} \right|_{\lambda=0} + \theta(0, \delta). \end{aligned} \quad (4.7)$$

Further,

$$\begin{aligned} \left. \frac{\partial \theta(\lambda, \delta)}{\partial \lambda} \right|_{\lambda=0} \approx & - \left(O^2 \Psi(1) \right)^T C \left(O^2 \Psi(\delta) \right) \\ & - \delta \left(\mu_2(1) - \mu_2(0) - \mu'_2(0) \right) \\ & - \left(\mu_1(1) - \mu_1(0) - \mu'_1(0) \right) \\ & - v_0(\delta) + v_1(\delta) = F(\delta). \end{aligned} \quad (4.8)$$

Substituting (4.8) in (4.6), we get

$$\begin{aligned} \theta(\lambda, \delta) \approx & \left(O^2 \Psi(\lambda) \right)^T C \left(O^2 \Psi(\delta) \right) \\ & + \delta \left(\mu_2(\lambda) - \mu_2(0) - \lambda \mu'_2(0) \right) \\ & + \left(\mu_1(\lambda) - \mu_1(0) - \lambda \mu'_1(0) \right) \\ & + \lambda F(\delta) + v_0(\delta). \end{aligned} \quad (4.9)$$

Employing fractional derivative on (4.9) both sides, we obtain

$$\begin{aligned} \frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} \approx & \left(O^2 \Psi(\lambda) \right)^T C \left(O^{2-\alpha} \Psi(\delta) \right) \\ & + \lambda D^\alpha F(\delta) + D^\alpha v_0(\delta), \end{aligned} \quad (4.10)$$

$$\begin{aligned} \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} \approx & \left(O^2 \Psi(\lambda) \right)^T C \left(O^{3-\alpha} \Psi(\delta) \right) \\ & + \left(\mu_2(\lambda) - \mu_2(0) - \lambda \mu'_2(0) \right) \frac{\Gamma(2)}{\Gamma(3-\alpha)} \delta^{2-\alpha} \\ & + \lambda D^{\alpha-1} F(\delta) + D^{\alpha-1} v_0(\delta), \end{aligned} \quad (4.11)$$

where

$$\begin{aligned} D^\alpha F(\delta) = & - \left(O^2 \Psi(1) \right)^T C \left(O^{2-\alpha} \Psi(\delta) \right) \\ & - D^\alpha v_0(\delta) + D^\alpha v_1(\delta) \end{aligned} \quad (4.12)$$

$$\begin{aligned} D^{\alpha-1} F(\delta) = & - \left(O^2 \Psi(1) \right)^T C \left(O^{3-\alpha} \Psi(\delta) \right) \\ & - \left(\mu_2(1) - \mu_2(0) - \mu'_2(0) \right) \frac{\Gamma(2)}{\Gamma(3-\alpha)} \delta^{2-\alpha} \\ & - D^{\alpha-1} v_0(\delta) + D^{\alpha-1} v_1(\delta). \end{aligned} \quad (4.13)$$

Substituting (4.5), (4.9), (4.10), and (4.11) in (4.1) at the collocation points (2.14), we get the following algebraic

equations

$$\begin{aligned} & \left(O^2 \Psi(\lambda) \right)^T C \left(O^{2-\alpha} \Psi(\delta) \right) + \lambda D^\alpha F(\delta) + D^\alpha v_0(\delta) \\ & + \left(O^2 \Psi(\lambda) \right)^T C \left(O^{3-\alpha} \Psi(\delta) \right) + \lambda D^{\alpha-1} F(\delta) \\ & + \left(\mu_2(\lambda) - \mu_2(0) - \lambda \mu'_2(0) \right) \frac{\Gamma(2)}{\Gamma(3-\alpha)} \delta^{2-\alpha} + D^{\alpha-1} v_0(\delta) \\ & + \left(O^2 \Psi(\lambda) \right)^T C \left(O^2 \Psi(\delta) \right) + \lambda F(\delta) + v_0(\delta) \\ & + \delta \left(\mu_2(\lambda) - \mu_2(0) - \lambda \mu'_2(0) \right) \\ & + \left(\mu_1(\lambda) - \mu_1(0) - \lambda \mu'_1(0) \right) - \Psi^T(\lambda) C \left(O^2 \Psi(\delta) \right) \\ & - \delta \mu''_2(\lambda) - \mu''_1(\lambda) \\ & = f(\lambda, \delta). \end{aligned} \quad (4.14)$$

By solving (4.14) we get undefined Bell wavelet coefficient vector C , and then by substituting C in (4.9) the approximate solution of (4.1) is obtained.

5 Convergence analysis

The fundamental findings regarding the approximation of Bell polynomials serve as the foundation for exploring the convergence of Bell wavelets approximations as:

For any function $\theta(\lambda) \in L^2[0, 1]$, and let $\mathcal{B}(\lambda) = [\mathcal{B}_0(\lambda), \mathcal{B}_1(\lambda), \dots, \mathcal{B}_n(\lambda)]$ can be expressed as

$$\theta(\lambda) \approx \theta_n(\lambda) = \sum_{i=0}^n c_i \mathcal{B}_i(\lambda). \quad (5.1)$$

The proof of Bell polynomials convergence is outlined in Lemma 5.1.

Lemma 5.1 [27] Consider $\theta(\lambda) \in L^2[0, 1]$, where $\theta_n(\lambda)$ defined in (5.1) is the best approximation of the real function $\theta(\lambda)$ by the Bell polynomials. Therefore, there $\exists$ a constant R such that

$$\|\theta(\lambda) - \theta_n(\lambda)\|_2 \leq \frac{R}{(n+1)!2^{2n+1}}.$$

Here $R = \max_{\lambda \in [0, 1]} |\theta^{n+1}(\lambda)|$.

Proof One can see the proof in [27]. $\square$

Remark 5.1 From Lemma 5.1, if $n \rightarrow \infty$, then $\frac{1}{(n+1)!2^{2n+1}} \rightarrow 0$, which means $\theta_n(\lambda) \rightarrow \theta(\lambda)$.

Theorem 5.1 [26] Suppose the Bell wavelets expansion $\sum_{m=1}^{2^{k-1}} \sum_{n=0}^{M-1} c_{mn} \psi_{mn}(\lambda) = \sum_{j=0}^{\hat{m}} c_j \psi_j(\lambda) = C^T \Psi(\lambda)$ of

the smooth function $\theta(\lambda)$, therefore

$$\lim_{\hat{m} \rightarrow \infty} \left\| \theta(\lambda) - \sum_{i=0}^{\hat{m}} c_i \psi_i(\lambda) \right\| = 0,$$

where $\hat{m} = 2^{k-1}M$.

Algorithm to solve time-fractional telegraph equation by Bell wavelet method

Algorithm 1

Input: $0 \leq \lambda \leq 1$, $k \in \mathbb{N}$ and $n = 0, 1, \dots, M-1$.

1: Define $\psi_{m,n}(\lambda)$ (Bell wavelet) using (2.4).

2: Calculate the vector $\Psi(\lambda)$ from (2.9).

3: Calculate the Bell wavelet basic matrix $\psi(\lambda)$ using collocation point (2.10).

4: Compute the OMI of order $2^{k-1}M \times 2^{k-1}M$ (3.14) using BPF (3.1).

5: Define unknown coefficient vector of Bell wavelet C in (2.8).

6: Calculate the (4.1) problem by approximating the highest time derivative in (4.4).

7: Solve the algebraic equation (4.14) using the iterative method in MATLAB.

8: Substitute the obtained undefined coefficient of Bell wavelet in (4.9).

Output: On $[0, 1]$, the approximate solution $\theta(\lambda, \delta)$ is obtained.

6 Numerical discussions

In this section, we will utilize the Bell wavelets and their fractional OMI to solve equation (1.1). To show the effectiveness of the algorithm, four test problems are considered with exact solutions. The results are then compared and shown geometrically. All the computational work is performed in MATLAB (R2023a).

Error estimation: We utilize the definition of absolute error, which is defined as,

$$E_j(z) = |\theta(\lambda, \delta) - \tilde{\theta}(\lambda, \delta)|,$$

to compare the approximate and exact solutions of the given equation at a few chosen points, where the approximate solution is given by $\tilde{\theta}(\lambda, \delta)$ and $\theta(\lambda, \delta)$ is the exact solution.

Problem 6.1 Let the time-fractional telegraph equation be:

$$\frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + \theta(\lambda, \delta) = \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \quad (6.1)$$

with ICs

$$\theta(\lambda, 0) = \lambda^2, \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = 1 + \lambda^2, \quad 0 \leq \lambda \leq 1, \quad (6.2)$$

and DBCs

$$\theta(0, \delta) = \delta, \quad \theta(1, \delta) = 1 + \delta, \quad 0 < \delta \leq 1, \quad (6.3)$$

where

$$f(\lambda, \delta) = \left(\frac{\Gamma(2)}{\Gamma(3-\alpha)} \right) + \lambda^2 + \delta - 2. \quad (6.4)$$

The exact solution of (6.1) is $f(\lambda, \delta) = \lambda^2 + \delta$. At $\alpha = 2$, Fig. 1 illustrates a side-by-side comparison between the exact solution and the Bell wavelet approximate solution, computed at resolution level $k = 3$, $M = 4$. This figure visually represents the approximation's accuracy and includes an error analysis that measures the deviation between the exact and approximate solutions. The error analysis highlights the effectiveness of the Bell wavelet method in closely approximating the exact solution under the specified parameters. To present the precision of the proposed methodology, the outcomes of L_2 and L_∞ are compared with the Sinc-Legendre method [29], Legendre method [28], and Fibonacci wavelet method [30] in Table 1. Table 1 clearly depicts that the results from the Bell wavelet are more efficient than the existing methods.

Problem 6.2 Consider the following equation

$$\frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + \theta(\lambda, \delta) = \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \quad (6.5)$$

with ICs

$$\theta(\lambda, 0) = 0, \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = 0, \quad 0 \leq \lambda \leq 1, \quad (6.6)$$

and DBCs

$$\theta(0, \delta) = \delta^{2\alpha}, \quad \theta(1, \delta) = \cos(7)\delta^{2\alpha}, \quad 0 < \delta \leq 1, \quad (6.7)$$

where

$$f(\lambda, \delta) = \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \delta^\alpha + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+2)} \delta^{\alpha+1} + 50\delta^{2\alpha} \right) \times \cos(7\lambda). \quad (6.8)$$

The exact solution of (6.5) is $f(\lambda, \delta) = \cos(7\lambda)\delta^{2\alpha}$. At $k = 3$, $M = 4$ resolution level, the approximate and exact solutions 3D graph are shown in Fig. 2 along with the error analysis. The Bell wavelet method approximate results at $\alpha = 1.1$ and 1.7 are presented in Fig. 3. These figures demonstrate how the method performs for different values of the fractional order, showcasing the flexibility and effectiveness of the Bell wavelet approach across a range of conditions. The comparison in 3D helps visualize the impact of varying fractional orders on the solution, providing insight into the method's robustness and accuracy in different scenarios.

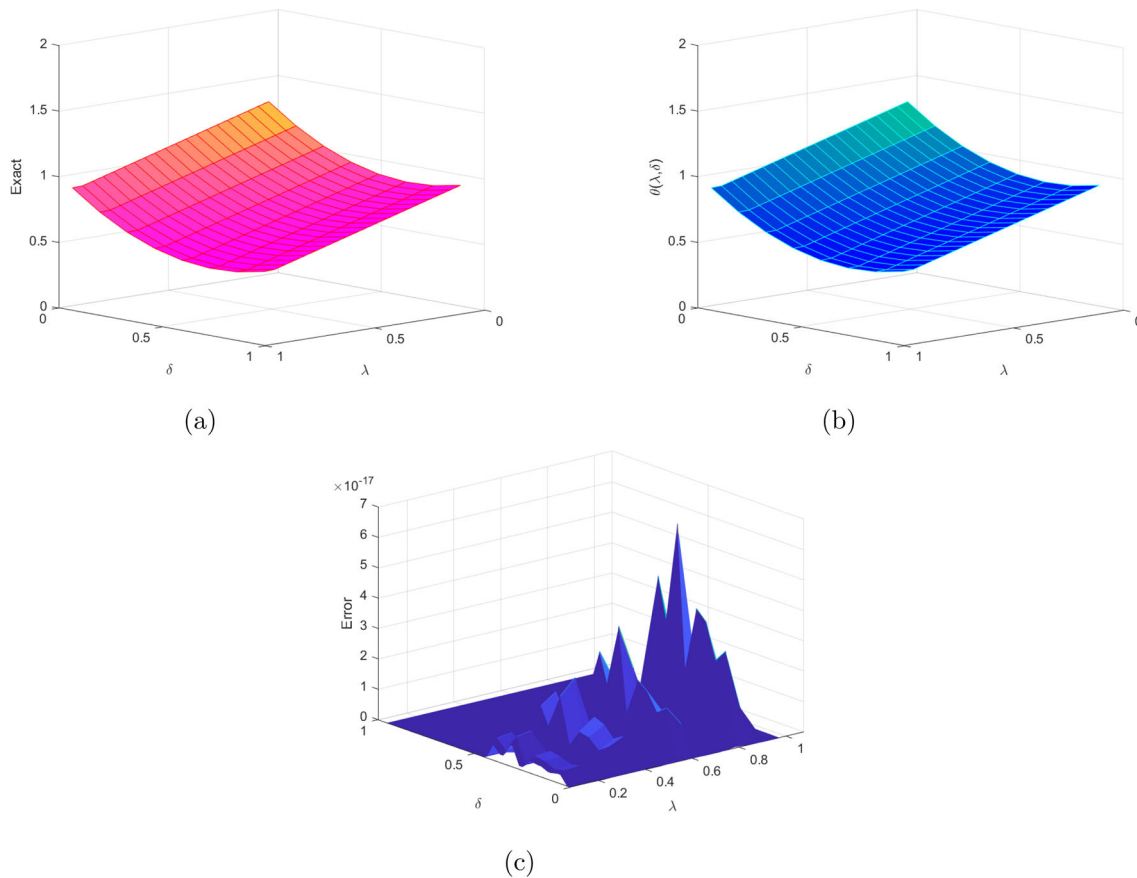


Fig. 1 a, b Exact and approximate solutions respectively and c Absolute error at $\alpha = 2$ of Problem 6.1. (CPU time: 1.595 sec)

Table 1 L_2, L_∞ error comparisons at $\alpha = 2$ of Problem 6.1

λ	Sinc-Legendre [29]		Legendre wavelet [28]		Fibonacci wavelet [30]		Present Method	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	8.64×10^{-4}	4.90×10^{-3}	7.5375×10^{-17}	2.2204×10^{-16}	3.2963×10^{-18}	4.2450×10^{-17}	6.5052×10^{-19}	5.8608×10^{-18}
0.3	8.06×10^{-4}	1.47×10^{-3}	5.5511×10^{-17}	1.1102×10^{-16}	1.1226×10^{-18}	6.1649×10^{-17}	1.7347×10^{-18}	3.5776×10^{-18}
0.5	1.38×10^{-3}	2.28×10^{-3}	7.8504×10^{-17}	1.1102×10^{-16}	0	0	3.4694×10^{-18}	5.9038×10^{-18}
0.7	7.50×10^{-4}	1.17×10^{-3}	1.2412×10^{-16}	2.2204×10^{-16}	0	0	6.9389×10^{-18}	3.9851×10^{-18}
0.9	8.60×10^{-5}	6.45×10^{-3}	2.0014×10^{-16}	4.4408×10^{-16}	3.5528×10^{-18}	3.7716×10^{-17}	0	0
1	0	0	2.0269×10^{-16}	2.2204×10^{-16}	0	0	0	0

Table 2 analyzes the comparison of numerical solutions with the Legendre wavelet [28] and Taylor wavelet [31] with the present method for $\alpha = 1.95$ and 1.25 . The results obtained from the Bell wavelet are better than the existing methods.

Problem 6.3 Consider the following equation

$$\frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + \theta(\lambda, \delta) = \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \quad (6.9)$$

with ICs

$$\theta(\lambda, 0) = 0, \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = \lambda(\lambda - 1), \quad 0 \leq \lambda \leq 1, \quad (6.10)$$

and DBCs

$$\theta(0, \delta) = 0, \quad \theta(1, \delta) = 0, \quad 0 < \delta \leq 1, \quad (6.11)$$

where

$$f(\lambda, \delta) = \left(\frac{\Gamma(2)}{\Gamma(3-\alpha)} \delta^{2-\alpha} \right) \lambda(\lambda - 1) - 2\delta. \quad (6.12)$$

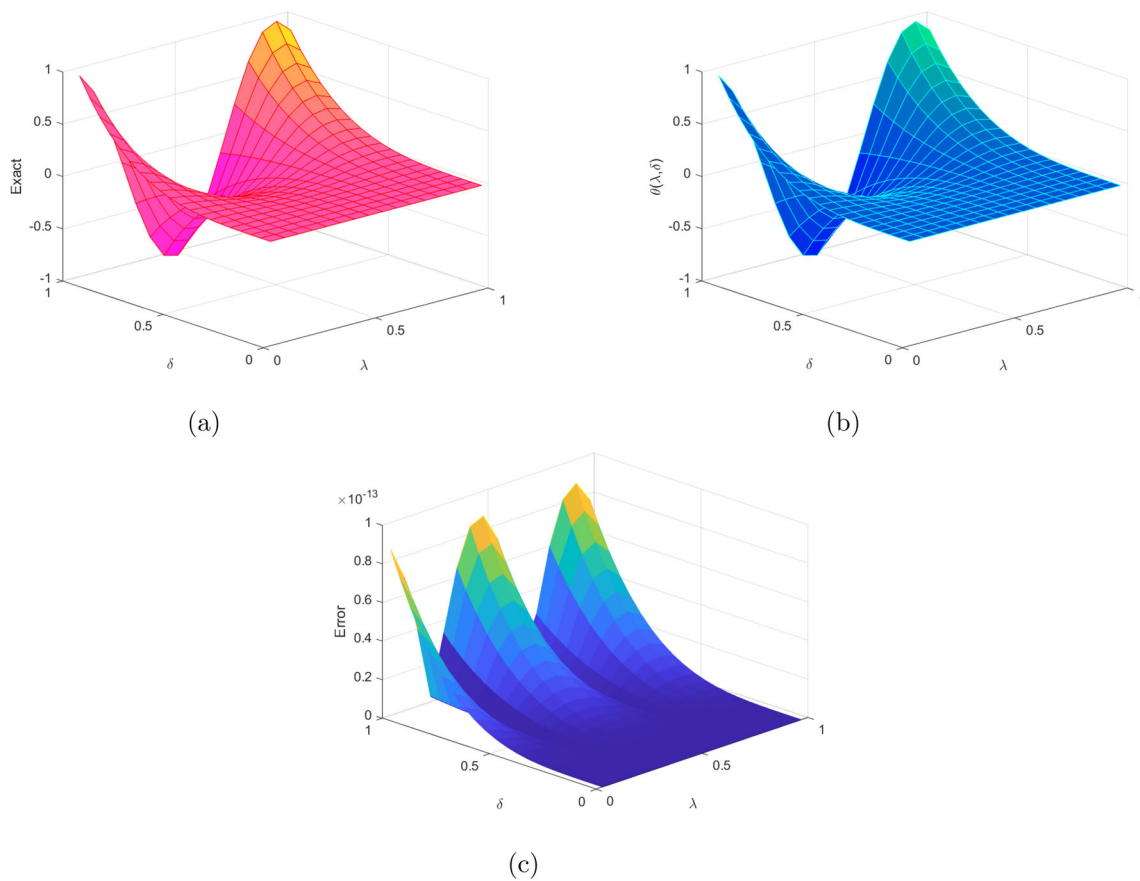


Fig. 2 **a, b** Exact and approximate solutions, respectively, and **c** absolute error at $\alpha = 1.95$ of Problem 6.2. (CPU time: 1.889 sec)

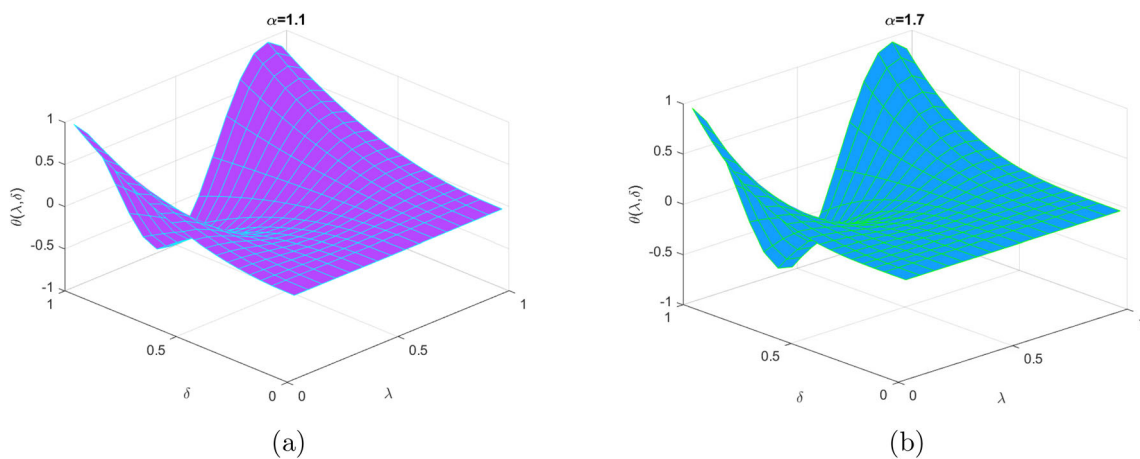


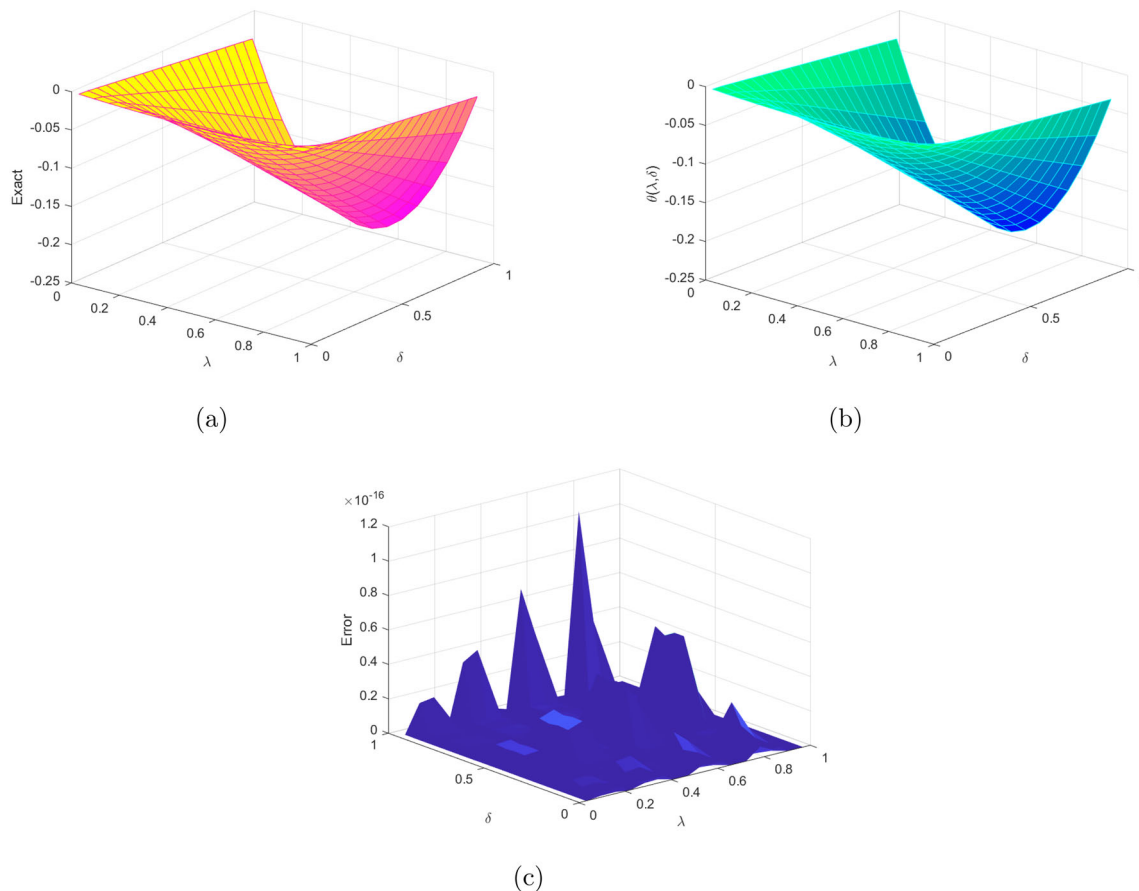
Fig. 3 Approximate solutions at $k = 3$, $M = 4$ for different α of Problem 6.2

The exact solution of (6.9) is $f(\lambda, \delta) = \lambda(\lambda - 1)\delta$. Figure 4 shows the graphical representation of the approximate and exact solutions along with the error analysis at resolution level $M = 4$, $k = 3$ for $\alpha = 1.9$. The graph clearly shows how the Bell wavelet method approximates the solution, and the accompanying error analysis highlights the accuracy of the method by comparing the approximate solution to the

exact one. The proposed Bell wavelet method is compared with the Sinc-Legendre method [29], Legendre method [28], Fibonacci wavelet method [30], and Taylor wavelet method [31] at $\delta=1$ in Table 3. The approximate results of the Bell wavelet method at $\alpha = 1.1, 1.3, 1.5$, and 1.7 are presented in Fig. 5. These plots illustrate how the method performs across a range of fractional orders, providing insights into

Table 2 Absolute error comparisons at $\alpha = 1.95, 1.25$ of Problem 6.2

(λ, δ)	$\alpha = 1.95$			$\alpha = 1.25$		
	Legendre wavelet [28]	Taylor wavelet [31]	Present method	Legendre wavelet [28]	Taylor wavelet [31]	Present method
(0.1,0.1)	3.8347×10^{-7}	7.9591×10^{-12}	1.5538×10^{-13}	6.5614×10^{-6}	7.4702×10^{-11}	1.0519×10^{-11}
(0.2,0.2)	2.3887×10^{-6}	2.2330×10^{-11}	7.3086×10^{-14}	4.1994×10^{-6}	3.0065×10^{-10}	4.9477×10^{-12}
(0.3,0.3)	4.4284×10^{-6}	6.9827×10^{-11}	6.2931×10^{-15}	1.1611×10^{-6}	6.8052×10^{-10}	4.1759×10^{-12}
(0.4,0.4)	4.7642×10^{-6}	1.0527×10^{-10}	6.1685×10^{-14}	6.9869×10^{-6}	1.2169×10^{-9}	7.9913×10^{-12}
(0.5,0.5)	1.9802×10^{-6}	1.9360×10^{-10}	1.5217×10^{-13}	9.0325×10^{-5}	1.9126×10^{-9}	1.0301×10^{-11}
(0.6,0.6)	2.0201×10^{-7}	2.6868×10^{-10}	1.339×10^{-13}	8.3791×10^{-5}	2.7705×10^{-9}	1.0671×10^{-11}
(0.7,0.7)	1.8224×10^{-7}	5.0108×10^{-10}	1.880×10^{-14}	6.2152×10^{-5}	3.7937×10^{-9}	3.3780×10^{-12}
(0.8,0.8)	2.8284×10^{-6}	6.5805×10^{-10}	1.0924×10^{-13}	4.4516×10^{-5}	4.9856×10^{-9}	1.0019×10^{-11}
(0.9,0.9)	1.5856×10^{-5}	9.0351×10^{-10}	1.3984×10^{-13}	5.5285×10^{-5}	6.3493×10^{-9}	1.0756×10^{-11}

**Fig. 4** (a), (b) Exact and approximate solutions, respectively, and (c) absolute error at $\alpha = 1.9$ of Problem 6.3. (CPU time:1.740 sec)

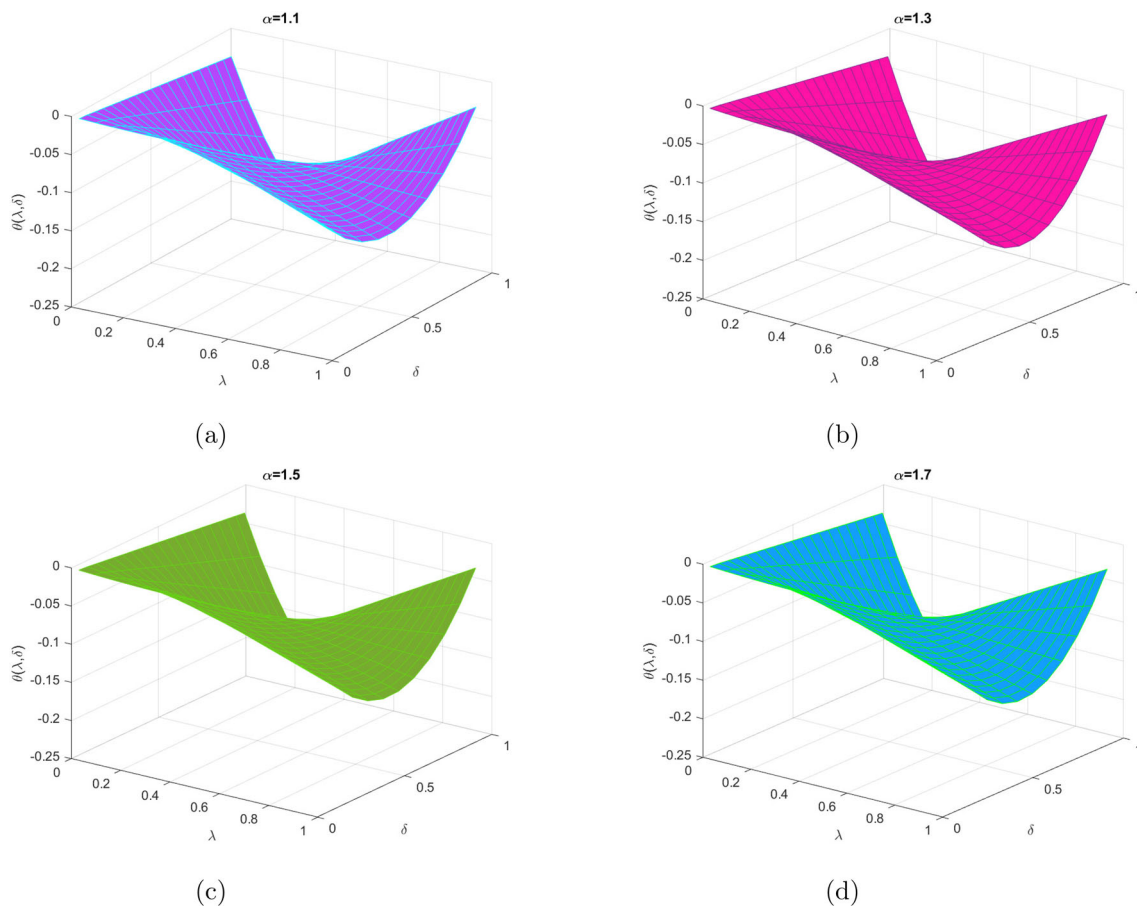
the method's behavior and accuracy for varying values of α . Table 4 shows the comparison of absolute error of the proposed method with the Legendre wavelet [28] and the Fibonacci wavelet [30] for $\alpha = 1.3$ and 1.5. The obtained outcomes show that the proposed Bell wavelet method provides more accurate and precise solutions than the [28–31].

Problem 6.4 Consider the following equation

$$\begin{aligned} \frac{\partial^\alpha \theta(\lambda, \delta)}{\partial \delta^\alpha} + \frac{\partial^{\alpha-1} \theta(\lambda, \delta)}{\partial \delta^{\alpha-1}} + \theta(\lambda, \delta) \\ = \pi \frac{\partial^2 \theta(\lambda, \delta)}{\partial \lambda^2} + f(\lambda, \delta), \quad 0 \leq \lambda \leq 1, \end{aligned} \quad (6.13)$$

Table 3 Absolute error comparisons of Problem 6.3

λ	Sinc-Legendre [29]	Legendre wavelet [28]	Fibonacci wavelet [30]		Taylor wavelet [31]		Present method
	$n = 3, m = 7$	$k = 2, M = 4$	$k = 2, M = 3$	$k = 2, M = 4$	$k = 1, M = 3$	$k = 2, M = 4$	
0.1	1.6932×10^{-4}	1.3877×10^{-17}	8.3267×10^{-17}	1.0714×10^{-18}	1.7347×10^{-18}	8.6736×10^{-19}	6.7763×10^{-21}
0.2	1.0916×10^{-3}	2.7755×10^{-17}	2.2204×10^{-16}	2.2204×10^{-18}	0	1.7347×10^{-18}	5.7175×10^{-21}
0.3	1.0749×10^{-3}	5.5511×10^{-17}	4.2896×10^{-15}	2.1957×10^{-18}	6.9388×10^{-18}	0	1.5416×10^{-19}
0.4	1.0102×10^{-3}	2.7755×10^{-17}	2.7756×10^{-17}	1.5266×10^{-17}	0	1.7347×10^{-18}	2.1176×10^{-20}
0.5	9.9270×10^{-4}	2.7755×10^{-17}	2.7756×10^{-17}	2.4182×10^{-17}	6.9388×10^{-18}	0	2.6258×10^{-20}
0.6	1.0102×10^{-3}	8.3266×10^{-17}	2.7756×10^{-17}	1.9429×10^{-18}	1.3877×10^{-17}	1.7347×10^{-18}	6.3527×10^{-21}
0.7	1.0749×10^{-3}	8.3266×10^{-17}	1.3878×10^{-16}	1.8041×10^{-18}	1.7347×10^{-17}	5.2041×10^{-18}	6.6092×10^{-18}
0.8	1.0916×10^{-3}	5.5511×10^{-17}	1.3878×10^{-16}	2.5258×10^{-17}	3.4694×10^{-18}	0	2.4352×10^{-21}
0.9	7.6032×10^{-4}	5.5511×10^{-17}	9.7145×10^{-17}	4.1737×10^{-17}	1.2143×10^{-17}	6.0715×10^{-18}	2.6470×10^{-22}

**Fig. 5** Approximate solutions at $k = 3, M = 4$ for different α of Problem 6.3

with ICs

$$\theta(\lambda, 0) = 0, \quad \frac{\partial \theta(\lambda, 0)}{\partial \delta} = 0, \quad 0 \leq \lambda \leq 1, \quad (6.14)$$

and DBCs

$$\theta(0, \delta) = 0, \quad \theta(1, \delta) = \delta^3 \sin(1), \quad 0 < \delta \leq 1, \quad (6.15)$$

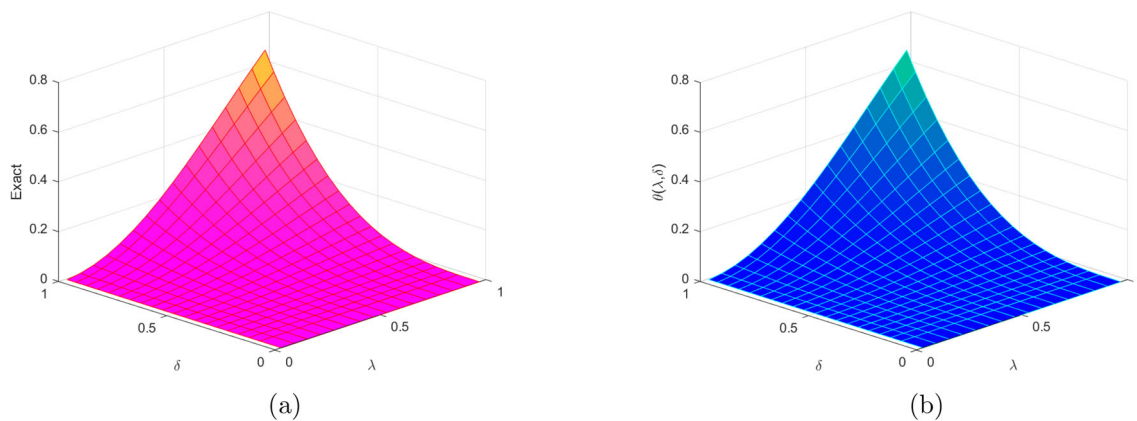
where

$$f(\lambda, \delta) = \left(\frac{3\Gamma(3)}{\Gamma(4-\alpha)} \delta^{3-\alpha} + \frac{3\Gamma(3)}{\Gamma(5-\alpha)} \delta^{4-\alpha} + \delta^3 \right) \times \sin^2(\lambda) - 2\pi \delta^3 \cos(2\lambda). \quad (6.16)$$

The exact solution of (6.13) is $f(\lambda, \delta) = \sin^2(\lambda) \delta^3$. The exact solution and Bell wavelet approximate solutions 3D

Table 4 Absolute error comparisons at $\alpha = 1.3, 1.5$ of Problem 6.3

(λ, δ)	$\alpha = 1.3$			$\alpha = 1.5$		
	Legendre wavelet [28]	Fibonacci wavelet [30]	Present method	Legendre wavelet [28]	Fibonacci wavelet [30]	Present method
(0.1,0.1)	1.73472×10^{-18}	6.6353×10^{-18}	5.8234×10^{-22}	1.73472×10^{-18}	1.9741×10^{-18}	2.6893×10^{-20}
(0.2,0.2)	6.93889×10^{-18}	4.5797×10^{-18}	9.7197×10^{-20}	0	4.3715×10^{-18}	7.6233×10^{-21}
(0.3,0.3)	0	4.6491×10^{-18}	8.4703×10^{-22}	6.9389×10^{-18}	8.6736×10^{-18}	5.9716×10^{-20}
(0.4,0.4)	1.38778×10^{-17}	0	1.3976×10^{-20}	1.38778×10^{-17}	6.6475×10^{-18}	2.4564×10^{-20}
(0.5,0.5)	0	7.6328×10^{-18}	1.2705×10^{-21}	0	6.7446×10^{-18}	1.6677×10^{-17}
(0.6,0.6)	2.77556×10^{-17}	7.9103×10^{-18}	2.1176×10^{-21}	2.77556×10^{-17}	6.2172×10^{-18}	7.4115×10^{-20}
(0.7,0.7)	5.55112×10^{-17}	2.6645×10^{-18}	8.4703×10^{-22}	2.77556×10^{-17}	1.4433×10^{-18}	2.5411×10^{-20}
(0.8,0.8)	1.11022×10^{-16}	1.1142×10^{-18}	3.1764×10^{-22}	8.32667×10^{-17}	3.3307×10^{-18}	7.0939×10^{-20}
(0.9,0.9)	5.55112×10^{-17}	1.2490×10^{-18}	1.0588×10^{-22}	4.16334×10^{-17}	2.3453×10^{-18}	1.8158×10^{-20}

**Fig. 6** (a), (b) Exact and approximate solutions respectively for resolution $k = 2, M = 4$ at $\alpha = 1.95$ of Problem 6.4. (CPU time: 1.656 sec)**Table 5** Absolute error comparisons at different α and M of Problem 6.4

λ	Legendre wavelet [28]			Fibonacci wavelet [30]			Present Method	
	$M = 3$	$M = 4$	$M = 5$	$M = 3$	$M = 4$	$M = 5$	$M = 3$	$M = 5$
1.15	4.6439×10^{-5}	7.5593×10^{-6}	2.0881×10^{-7}	4.1184×10^{-5}	6.3146×10^{-6}	1.6982×10^{-8}	5.8790×10^{-6}	2.1095×10^{-9}
1.35	4.5474×10^{-5}	7.3935×10^{-6}	2.0391×10^{-7}	3.0138×10^{-5}	6.4611×10^{-6}	1.3261×10^{-8}	4.3886×10^{-6}	1.9776×10^{-9}
1.55	4.4293×10^{-5}	7.1980×10^{-6}	1.9830×10^{-7}	3.0138×10^{-5}	5.8519×10^{-6}	2.8519×10^{-8}	4.3427×10^{-5}	1.8365×10^{-9}
1.75	4.2909×10^{-5}	6.9749×10^{-6}	1.9191×10^{-7}	4.2956×10^{-5}	6.8947×10^{-6}	1.4617×10^{-8}	4.1297×10^{-5}	1.8019×10^{-9}
1.95	4.1016×10^{-5}	6.6688×10^{-6}	1.8310×10^{-7}	4.1642×10^{-5}	6.2791×10^{-6}	1.9431×10^{-8}	4.0961×10^{-5}	1.7847×10^{-9}

graphs are shown in Fig. 6 at $\alpha = 1.95$ for resolution level $k = 2, M = 4$. The 3D graphs illustrate how the Bell wavelet method approximates the solution to the fractional telegraph equation by visualizing the surface of both the exact solution and the wavelet-based approximation at the given values of α, k, M . Table 5 presents the absolute error comparison for different resolutions $M = 3, 4$, and 5 at $\alpha = 1.15, 1.35, 1.55, 1.75$, and 1.95 . The Bell wavelet method's approximate results are then compared with the Legendre [28] and the Fibonacci wavelet method [30]. Table 5 clearly shows that the proposed method gives high accuracy and precise results.

7 Conclusion

In this study, we formulated an efficient hybrid method utilizing the Bell wavelet and BPFs to solve the time-fractional telegraph equations. Bell wavelets and their corresponding OMIs are constructed the first time by using BPFs. To demonstrate the effectiveness of the Bell wavelet technique, four distinct test problems have been examined. Figures 1a, 1b, 2a, 2b, 4a, 4b, 6a, and 6b present the 3D graphs of the exact and approximate solutions. The approximate solutions at various α are depicted in Figs. 3a, 3a, 5a, 5b, 5c, and 5d. To demonstrate the proposed method's accuracy, the compar-

ison of absolute errors is shown in Tables 1, 2, 3, and 4 at different α . The results are compared with the Sinc-Legendre method [29], Fibonacci wavelet [30], Legendre wavelet [28], and Taylor wavelet [31] methods. The results in Table 5 show that increasing the resolution leads to a more accurate solution. Therefore, it is clearly depicted that the proposed methodology is straightforward, highly robust, and effective in dealing with fractional partial differential equations. In future, this proposed method can be extended to solve higher-dimensional problems, broadening its applicability. It holds potential for use in diverse engineering and biological models, making it particularly suited for tackling dispersion equations, heat and diffusion equations, and equations governing linear electric circuits.

Acknowledgements We would like to thank Central University of Haryana for providing the necessary facilities to carry out this research.

Data availability No data was used for the research described in the article.

Code availability Code may be provided on the request.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

- Hunt BJ (1994) *The Maxwellians*. Cornell University Press
- Holmes EE (1993) Are diffusion models too simple? A comparison with telegraph models of invasion. *The American Naturalist*. 142(5):779–95
- Pascal H (1986) Pressure wave propagation in a fluid flowing through a porous medium and problems related to interpretation of Stoneley's wave attenuation in acoustical well logging. *Int J Eng sci*. 24(9):1553–70
- Böhme G (2012) *Non-Newtonian fluid mechanics*. Elsevier;
- Jordan PM, Puri A (1999) Digital signal propagation in dispersive media. *J Appl Phys* 85(3):1273–82
- Debnath L (2005) *Nonlinear partial differential equations for scientists and engineers*. Birkhäuser, Boston
- Barletta A, Zanchini E (1999) A thermal potential formulation of hyperbolic heat conduction. *J heat transfer*. 121(1):166–9
- Singh H (2017) A new stable algorithm for fractional Navier-Stokes equation in polar coordinate. *Int J Appl Computational Mathematics*. 3(4):3705–22
- Singh H (2016) A new numerical algorithm for fractional model of Bloch equation in nuclear magnetic resonance. *Alexandria Eng J* 55(3):2863–9
- Bagley RL, Torvik PJ (1983) Fractional calculus-a different approach to the analysis of viscoelastically damped structures. *AIAA J* 21(5):741–8
- Panda R, Dash M (2006) Fractional generalized splines and signal processing. *Signal Processing*. 86(9):2340–50
- Lin Y, Xu C (2007) Finite difference/spectral approximations for the time-fractional diffusion equation. *J computational phys* 225(2):1533–52
- Deng W (2009) Finite element method for the space and time fractional Fokker-Planck equation. *SIAM J numerical analysis*. 47(1):204–26
- Meerschaert MM, Scheffler HP, Tadjeran C (2006) Finite difference methods for two-dimensional fractional dispersion equation. *J Computational phys* 211(1):249–61
- Diethelm K, Ford J (2002) Numerical solution of the Bagley-Torvik equation. *BIT numerical mathematics*. 42:490–507
- Diethelm K, Ford NJ, Freed AD, Luchko Y (2005) Algorithms for the fractional calculus: a selection of numerical methods. *Computer methods in Appl Mechanics Eng* 194(6–8):743–73
- Momani S (2005) Analytic and approximate solutions of the space- and time-fractional telegraph equations. *Appl Mathe Computation*. 170(2):1126–34
- Chen J, Liu F, Anh V (2008) Analytical solution for the time-fractional telegraph equation by the method of separating variables. *J Mathematical Anal Appl* 338(2):1364–77
- Sweilam NH, Nagy AM, El-Sayed AA (2016) Solving time-fractional order telegraph equation via Sinc-Legendre collocation method. *Mediterranean J Mathematics*. 13:5119–33
- Mohammadian S, Mahmoudi Y, Saei FD (2019) Solution of fractional telegraph equation with Riesz space-fractional derivative. *AIMS Mathematics*. 4(6):1664–83
- Modanli M, Akgül A (2020) On Solutions of Fractional order Telegraph partial differential equation by Crank-Nicholson finite difference method. *Appl Mathematics and Nonlinear Sci* 5(1):163–70
- Huang F (2009) Analytical solution for the time-fractional telegraph equation. *Journal of Applied Mathematics*. 2009;
- Das S, Gupta PK (2011) Homotopy analysis method for solving fractional hyperbolic partial differential equations. *Int J Computer Mathematics*. 88(3):578–88
- Yıldırım A (2010) He's homotopy perturbation method for solving the space- and time-fractional telegraph equations. *Int J of Computer Mathematics*. 87(13):2998–3006
- Singh H (2017) Operational matrix approach for approximate solution of fractional model of Bloch equation. *J of King Saud University-Sci* 29(2):235–40
- Wang Y, Zhou X (2024) An Efficient Numerical Method Based on Bell Wavelets for Solving the Fractional Integro-Differential Equations with Weakly Singular Kernels. *Fractal and Fractional*. 8(2):74
- Mirzaee F (2017) Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials. *Computational Methods for Differential Equations*. 5(2):88–102
- Xu X, Xu D (2018) Legendre wavelets direct method for the numerical solution of time-fractional order telegraph equations. *Mediterranean J Mathematics*. 15:1–33
- Sweilam NH, Nagy AM, El-Sayed AA (2016) Solving time-fractional order telegraph equation via Sinc-Legendre collocation method. *Mediterranean J Mathematics*. 13:5119–33
- Shah FA, Irfan M, Nisar KS, Matoog RT, Mahmoud EE (2021) Fibonacci wavelet method for solving time-fractional telegraph equations with Dirichlet boundary conditions. *Results in Phys* 24:104123
- Mulimani M (2024) Numerical solution of time-fractional telegraph equations using wavelet transform. *Int J Dynamics and Control*. 12(7):2166–89
- Masood S, Khan H, Shah R, Mustafa S, Khan Q, Arif M, Tchier F, Singh G (2022) A New Modified Technique of Adomian Decomposition Method for Fractional Diffusion Equations with Initial-Boundary Conditions. *J Function Spaces*. 1:6890517
- Ahmad S, Shah K, Abdeljawad T, Abdalla B (2023) On the Approximation of Fractal-Fractional Differential Equations Using Numerical Inverse Laplace Transform Methods. *CMES-Computer Modeling in Eng Sci* 135:3

34. Chawla R, Kumar D (2024) A High Order Numerical Scheme for Time-Fractional Telegraph Equation Via Cubic Spline in Tension. *Differential Equations and Dynamical Systems*. 1:26
35. Rayal A, Verma SR (2022) Two-dimensional Gegenbauer wavelets for the numerical solution of tempered fractional model of the non-linear Klein-Gordon equation. *Applied Numerical Mathematics*. 174:191–220
36. Delić A, Jovanović BS, Živanović S (2020) Finite difference approximation of a generalized time-fractional telegraph equation. *Computational Methods in Applied Mathematics*. 20(4):595–607
37. Choudhary R, Singh S, Kumar D (2024) A higher order unconditionally stable numerical technique for multi-term time-fractional diffusion and advection-diffusion equations. *Computational and Applied Mathematics*. 43(5):303
38. Rayal A, Verma SR (2021) An approximate wavelets solution to the class of variational problems with fractional order. *J Appl Mathematics and Computing*. 65(1):735–69
39. Paul S, Mahata A, Mukherjee S, Mali PC, Roy B (2023) Fractional order SEIQRD epidemic model of Covid-19: A case study of Italy. *PLoS One*. 18(3):e0278880
40. Paul S, Mahata A, Karak M, Mukherjee S, Biswas S, Roy B (2024) Dynamical behavior of fractal-fractional order monkeypox virus model. *Franklin Open*. 7:100103
41. Paul S, Mahata A, Mukherjee S, Das M, Mali PC, Roy B, Mukherjee P, Bharati P (2024) Study of fractional order SIR model with MH type treatment rate and its stability analysis. *Bulletin of Biomathematics*. 2(1):85–113
42. Paul S, Mahata A, Mukherjee S, Mali PC, Roy B (2022) Study of fractional order tri-tropic prey-predator model with fear effect on prey population. *Advances in Pure Mathematics*. 12(11):652–75
43. Paul S, Mahata A, Karak M, Mukherjee S, Biswas S, Roy B (2024) A fractal-fractional order Susceptible-Exposed-Infected-Recovered (SEIR) model with Caputo sense. *Healthcare Analytics*. 5:100317

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Terms and Conditions

Springer Nature journal content, brought to you courtesy of Springer Nature Customer Service Center GmbH (“Springer Nature”).

Springer Nature supports a reasonable amount of sharing of research papers by authors, subscribers and authorised users (“Users”), for small-scale personal, non-commercial use provided that all copyright, trade and service marks and other proprietary notices are maintained. By accessing, sharing, receiving or otherwise using the Springer Nature journal content you agree to these terms of use (“Terms”). For these purposes, Springer Nature considers academic use (by researchers and students) to be non-commercial.

These Terms are supplementary and will apply in addition to any applicable website terms and conditions, a relevant site licence or a personal subscription. These Terms will prevail over any conflict or ambiguity with regards to the relevant terms, a site licence or a personal subscription (to the extent of the conflict or ambiguity only). For Creative Commons-licensed articles, the terms of the Creative Commons license used will apply.

We collect and use personal data to provide access to the Springer Nature journal content. We may also use these personal data internally within ResearchGate and Springer Nature and as agreed share it, in an anonymised way, for purposes of tracking, analysis and reporting. We will not otherwise disclose your personal data outside the ResearchGate or the Springer Nature group of companies unless we have your permission as detailed in the Privacy Policy.

While Users may use the Springer Nature journal content for small scale, personal non-commercial use, it is important to note that Users may not:

1. use such content for the purpose of providing other users with access on a regular or large scale basis or as a means to circumvent access control;
2. use such content where to do so would be considered a criminal or statutory offence in any jurisdiction, or gives rise to civil liability, or is otherwise unlawful;
3. falsely or misleadingly imply or suggest endorsement, approval, sponsorship, or association unless explicitly agreed to by Springer Nature in writing;
4. use bots or other automated methods to access the content or redirect messages
5. override any security feature or exclusionary protocol; or
6. share the content in order to create substitute for Springer Nature products or services or a systematic database of Springer Nature journal content.

In line with the restriction against commercial use, Springer Nature does not permit the creation of a product or service that creates revenue, royalties, rent or income from our content or its inclusion as part of a paid for service or for other commercial gain. Springer Nature journal content cannot be used for inter-library loans and librarians may not upload Springer Nature journal content on a large scale into their, or any other, institutional repository.

These terms of use are reviewed regularly and may be amended at any time. Springer Nature is not obligated to publish any information or content on this website and may remove it or features or functionality at our sole discretion, at any time with or without notice. Springer Nature may revoke this licence to you at any time and remove access to any copies of the Springer Nature journal content which have been saved.

To the fullest extent permitted by law, Springer Nature makes no warranties, representations or guarantees to Users, either express or implied with respect to the Springer nature journal content and all parties disclaim and waive any implied warranties or warranties imposed by law, including merchantability or fitness for any particular purpose.

Please note that these rights do not automatically extend to content, data or other material published by Springer Nature that may be licensed from third parties.

If you would like to use or distribute our Springer Nature journal content to a wider audience or on a regular basis or in any other manner not expressly permitted by these Terms, please contact Springer Nature at

onlineservice@springernature.com